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The Multiple Scattering of Waves by Weak Random Irregularities in the Medium

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THE MULTIPLE SCATTERING OF WAVES BY WEAK RANDOM IRREGULARITIES IN THE MEDIUM

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CONTENTS

Papers by Lighthill (1953) and Fejer (1953) have treated multiple scattering by supposing a wave to be scattered any number of times in accordance with the cross-section for single scattering. This paper extends this idea, and uses the equation of energy transfer for radiation in a uniform scattering atmosphere to describe the variation of average intensity in a randomly inhomogeneous medium.

In part I, the results of the single-scattering theory are reviewed, and an estimate is made of the conditions under which they should be correct. The justification for the treatment of multiple scattering by an equation of energy transfer is then discussed, and conditions under which it may be expected to be valid are obtained.

In part II, the general solution of the equation of transfer for a spatially homogeneous radiation field, varying with time, is given first, and compared with Lighthill's result for the angular distribution of radiation in terms of the length of path travelled. The much more difficult problem of a steady-state field with spatial variation has been treated by Chandrasekhar (1950), who gives many exact solutions for special types of scattering (such as isotropic and Rayleigh scattering). But his methods are not well suited to some other types, especially small-angle forward scattering. Most of part II is devoted to finding approximate solutions for this case, first generalizing Fejer's solution for a slab of scattering medium which produces a small total angular deviation of the radiation, and then deriving an approximate partial differential equation of transfer to treat problems where the total angular deviation is not small. Methods of solving this equation by eigenfunction expansions are explained, and some numerical results are given, especially angular distributions of emergent and reflected radiation for a semi-infinite scattering region.

1. Introduction

The many papers which have dealt with the scattering of waves by random fluctuations in refractive index, in recent years, have nearly all confined their attention to scattering regions sufficiently small for the approximation of single scattering to be valid. But problems involving multiple scattering have been discussed by Fejer (1953) and Lighthill (1953).

These two authors make the assumption that the cross-section for scattering from a unidirectional beam of radiation, at a given angle to the direction of the beam, is the same whether the incident radiation is a coherent plane wave, or the result of other scatterings, and therefore incoherent. By considering average intensities of radiation they reduce the problem to that of radiation in a uniform scattering atmosphere, characterized by the rate of scattering, and the angular distribution of scattered energy—that is, by the scattering cross-section.

They consider the problem only for wavelengths small compared with the smallest length scale of the irregularities, so that only small angles of deviation occur at each scattering. Moreover, Fejer further confines his attention to relatively small thicknesses of the scattering region, such that the total angular deviation from the direction of the incident wave is small. Lighthill, on the other hand, discusses the manner in which the angular distribution of the radiation approaches isotropy, after very many scatterings, in terms of the path length travelled by the radiation.

The first part of the present paper sets out to examine more carefully the assumptions made (and the conditions under which they are valid) when the problem is reduced to that of radiation in a uniform scattering atmosphere, where each particle scatters energy in accordance with a fixed distribution, independently of all other particles. We begin (in § 2) by defining the quantities (intensity, energy flux, etc.) appropriate to the latter problem, and by giving the general equation of transfer which is satisfied. And we want to see under what conditions the same equation is applicable to waves in a randomly inhomogeneous medium, if the quantities in the equation are interpreted now as average values.

With this object, §§ 3 to 6 are devoted to an outline of the standard, single-scattering solution of the wave equation in a randomly inhomogeneous medium, and to an estimate of its range of validity, obtained by taking the iteration process one stage further. In § 7 the problem of giving a precise meaning to terms such as 'intensity in a given direction', in such a medium, is considered. In §§ 8 and 9 it is shown that it is legitimate, under certain conditions, to use the cross-section obtained from the single-scattering solution, for the scattering of the energy confined to a small solid angle of directions at any point in the medium, and hence to describe the variation of average intensity by the equation of energy transfer as derived in § 2.

The second part of the paper is concerned with solving the equation of transfer, as outlined in the summary.

I. Equation for the average intensity in multiple scattering of a wave by random irregularities

2. Preliminary definitions, and the equation of transfer

We consider an idealized situation in which monochromatic radiant energy travels in straight lines in all directions at constant speed, in a scattering atmosphere. Each element of volume scatters independently of all other elements. We define the following terms.

(i) *Intensity of radiation* is the rate of flow of energy travelling in a given direction, across a surface element normal to that direction, per unit area of surface, per unit solid angle of direction of radiation. It is a function of position $\mathbf{r}$, of time t, and of direction $\boldsymbol{\rho}$ (unit vector).

$$I = I(\mathbf{r}, t, \boldsymbol{\rho}).$$

(ii) *Net flux vector* has its component in direction $\boldsymbol{\rho}'$ equal to the net rate of flow of energy travelling in all directions, across a surface element with outward normal $\boldsymbol{\rho}'$, per unit area of surface. It is a function of position and time.

$$\pi\mathbf{F}(\mathbf{r}, t) . \boldsymbol{\rho}' = \int I(\mathbf{r}, t, \boldsymbol{\rho})\, \boldsymbol{\rho} . \boldsymbol{\rho}' \,\mathrm{d}\Omega$$
$$= \left\{\int I(\mathbf{r}, t, \boldsymbol{\rho})\, \boldsymbol{\rho}\, \mathrm{d}\Omega\right\} . \boldsymbol{\rho}',$$

integrated over all solid angles of $\boldsymbol{\rho}$. Hence

$$\pi\mathbf{F}(\mathbf{r}, t) = \int I(\mathbf{r}, t, \boldsymbol{\rho})\, \boldsymbol{\rho}\, \mathrm{d}\Omega.$$

$\pi F = \pi\, |\mathbf{F}|$ will be called simply the net flux.

(iii) *Velocity of propagation* of the radiation, c.

(iv) *Absorption coefficient* of the medium, $\beta(\boldsymbol{\rho})$, is the fraction of energy lost by scattering and dissipative processes (assumed linear) from a pencil of radiation in direction $\boldsymbol{\rho}$, per unit path length travelled in the medium.

(v) *Scattering function* of the medium, $p(\boldsymbol{\rho}, \boldsymbol{\rho}')$, specifies the angular distribution of scattered energy, in such a way that

$$(4\pi)^{-1} \beta(\boldsymbol{\rho})\, p(\boldsymbol{\rho}, \boldsymbol{\rho}')\, I(\mathbf{r}, t, \boldsymbol{\rho})\, \mathrm{d}\Omega\, \mathrm{d}V \mathrm{d}\Omega'$$

is the rate at which energy is scattered, from a pencil of radiation of intensity $I(\mathbf{r}, t, \boldsymbol{\rho})$ in a solid angle $\mathrm{d}\Omega$ in direction $\boldsymbol{\rho}$, by a volume $\mathrm{d}V$ of medium at position $\mathbf{r}$, into a solid angle $\mathrm{d}\Omega'$ in direction $\boldsymbol{\rho}'$. Thus $(4\pi)^{-1}\beta(\boldsymbol{\rho})\, p(\boldsymbol{\rho}, \boldsymbol{\rho}') = \sigma(\boldsymbol{\rho}, \boldsymbol{\rho}')$ is the cross-section for scattering from direction $\boldsymbol{\rho}$ to direction $\boldsymbol{\rho}'$.

Although this has not been indicated explicitly, β, p and σ may depend also on position, without affecting the equation to be derived in this section.

(vi) *Source function*, $J(\mathbf{r}, t, \boldsymbol{\rho})$ is defined so that $\beta J \,\mathrm{d}V \mathrm{d}\Omega$ is the rate at which energy is scattered (in a given radiation field) from a volume element $\mathrm{d}V$ at position $\mathbf{r}$, into solid angle $\mathrm{d}\Omega$ in direction $\boldsymbol{\rho}$.

$$\beta(\boldsymbol{\rho})\, J(\mathbf{r}, t, \boldsymbol{\rho}) = \frac{1}{4\pi}\int \beta(\boldsymbol{\rho}')\, p(\boldsymbol{\rho}', \boldsymbol{\rho})\, I(\mathbf{r}, t, \boldsymbol{\rho}')\, \mathrm{d}\Omega',$$

integrated over all solid angles of $\boldsymbol{\rho}'$.

It may be noted that (iii), (iv) and (v) depend only on the medium, while (i), (ii) and (vi) apply to a given radiation field.

Considering now the energy travelling in direction $\boldsymbol{\rho}$, within solid angle $\mathrm{d}\Omega$, in volume $\mathrm{d}V$, we find the *equation of transfer*

$$\frac{1}{c}\frac{\partial I}{\partial t} + \boldsymbol{\rho}\,.\,\nabla_{\mathbf{r}} I = \beta(J - I),$$

or,

$$\left\{\frac{1}{c}\frac{\partial}{\partial t} + \boldsymbol{\rho}\,.\,\nabla_{\mathbf{r}} + \beta(\boldsymbol{\rho})\right\} I(\mathbf{r}, t, \boldsymbol{\rho}) = \frac{1}{4\pi}\int \beta(\boldsymbol{\rho}')\, p(\boldsymbol{\rho}', \boldsymbol{\rho})\, I(\mathbf{r}, t, \boldsymbol{\rho}')\, \mathrm{d}\Omega'. \tag{1}$$

This is the equation which we wish to show can be applied to the problem of multiple scattering of waves by random fluctuations in the medium. It can be generalized to describe scattering processes which involve frequency changes by making the intensity a function of $\mathbf{r}$, t, and $\boldsymbol{\kappa}$, the wave vector, and taking the integral for the source function over all wave vectors. It can also be generalized to apply to radiation which is polarized, when the scattering cross-section depends on the direction of polarization (see Chandrasekhar 1950). But the treatment in the following sections does not apply to these cases.

3. Waves in a region of weak random irregularities

The situation to which we wish to apply the equation of transfer is that of waves in a medium having small random fluctuations in its properties. We shall refer to the following cases:

(*a*) Scattering of a sound wave by fluctuations in sound speed, the density being constant.

(*b*) Scattering of a sound wave by fluctuations in sound speed and density, the elasticity being constant (e.g. air with fluctuations of temperature or humidity).

(*c*) Scattering of a sound wave by turbulent motion of the medium.

(*d*) Scattering of an electromagnetic wave by fluctuations in permittivity.

Let the waves have frequency $\omega/2\pi$, and suppose that the time variation of the fluctuations can be neglected (this point is referred to in § 9). The sound wave is specified by the function $\phi(\mathbf{r})\,\mathrm{e}^{-\mathrm{i}\omega t}$, and in cases (*a*) and (*b*) the sound speed is $c = c_0(1 + \frac{1}{2}f(\mathbf{r}))$, where $f(\mathbf{r})$ is a small random function with zero mean, vanishing outside a suitably bounded region—the scattering volume. We suppose that the covariance of $f(\mathbf{r})$ can be written in the form

$$\overline{f(\mathbf{r})f(\mathbf{r}+\mathbf{s})} = \eta^2 B(\mathbf{r})\, C(\mathbf{s}) \tag{2}$$

(the overbar denotes the mean over all realizations of the random function $f(\mathbf{r})$). Strictly, for consistency, we should have $B(\mathbf{r}+\frac{1}{2}\mathbf{s})$ rather than $B(\mathbf{r})$ (Silverman 1958), but if B changes very little over distances for which the covariance is appreciable, the form assumed will be sufficiently accurate. It has the advantage of simplifying some of the calculations.

$B(\mathbf{r})$ has a maximum value of unity, and describes the variation of mean-square fluctuation over the scattering volume. $\int B(\mathbf{r})\,\mathrm{d}\mathbf{r}$, over all space, will be called the effective volume, V. When appropriate, we shall refer to an effective diameter, M, of the volume, such that $B(\mathbf{r}) \doteqdot 0$ if $r \gg M$.

η will be called the degree of irregularity of the scattering volume.

$C(\mathbf{s})$ is the correlation coefficient of the fluctuations, so that $C(0) = 1$. It can be expressed as the Fourier transform of the spectrum function $\Phi(\boldsymbol{\chi})$

$$C(\mathbf{s}) = \int e^{i\boldsymbol{\chi}.\mathbf{s}}\,\Phi(\boldsymbol{\chi})\,d\boldsymbol{\chi}.$$

For isotropic irregularities, C and Φ are functions only of the magnitudes s and χ, respectively, and $\int_0^\infty C(s)\,ds = L$ will be called the length scale of the irregularities. Otherwise, we shall take L to be the maximum value of $\int_0^\infty C(s\boldsymbol{\rho})\,ds$, over all directions of the unit vector $\boldsymbol{\rho}$.

We require that $M \gg L$.

In the other cases, (*c*) and (*d*), the notation is the same except for the definition of $f(\mathbf{r})$. In (*c*), the quantity responsible for the scattering is a vector, the turbulent velocity, but for single scattering from a plane wave only the component of velocity in the direction of propagation of the incident wave is relevant, and we define $f(\mathbf{r})$ as twice the Mach number of this component. In (*d*), $f(\mathbf{r})$ is the fractional fluctuation of permittivity from its mean value.

In case (*a*), the wave equation is

$$\nabla^2\phi + \kappa^2\phi = \kappa^2\phi f, \tag{3}$$

where $\kappa = \omega/c_0$, the wave number in the undisturbed medium. This can also be written

$$\phi(\mathbf{r}) = \phi_0(\mathbf{r}) - \frac{1}{4\pi}\kappa^2\int \exp\left[i\kappa\,|\mathbf{r}-\mathbf{r}'|\right]|\mathbf{r}-\mathbf{r}'|^{-1}f(\mathbf{r}')\,\phi(\mathbf{r}')\,d\mathbf{r}', \tag{4}$$

where ϕ_0 is a solution of equation (3) with $f = 0$, and can be called the incident wave. The other cases lead to equations similar to (3), but with different right-hand sides (Obukhoff 1943; Lighthill 1953; Kraichnan 1953; Batchelor 1957).

4. Results of the single-scattering approximation

For the single-scattering approximation, the standard method of solution when the scattering volume is not too large, we take the first approximation to the iteration solution of (4), substituting ϕ_0 for ϕ in the integral. Then for an incident plane wave, $\phi_0 = A\,e^{i\boldsymbol{\kappa}.\mathbf{r}}$, we have (still referring to case (*a*)),

$$\begin{aligned}\phi &= \phi_0 + \phi_1\\ &= A\exp\left[i\boldsymbol{\kappa}.\mathbf{r}\right] - \frac{1}{4\pi}A\kappa^2\int \exp\left[i\kappa\,|\mathbf{r}-\mathbf{r}'| + i\boldsymbol{\kappa}.\mathbf{r}'\right]|\mathbf{r}-\mathbf{r}'|^{-1}f(\mathbf{r}')\,d\mathbf{r}'.\end{aligned}$$

At points sufficiently far from the scatterer, we can further approximate:

$$\phi(\mathbf{r}) = A\,e^{i\boldsymbol{\kappa}.\mathbf{r}} - \frac{A\kappa^2}{4\pi r}\,e^{i\kappa r}\int e^{-i\boldsymbol{\chi}.\mathbf{r}'}f(\mathbf{r}')\,d\mathbf{r}', \tag{5}$$

where

$$\begin{aligned}\boldsymbol{\chi} &= \kappa\mathbf{r}/r - \boldsymbol{\kappa}\\ &= \kappa(\boldsymbol{\rho}' - \boldsymbol{\rho}).\end{aligned}$$

$\boldsymbol{\rho}$, $\boldsymbol{\rho}'$ are unit vectors in the directions of the incident and scattered waves (figure 1), since, to a good approximation at large distances, the scattered wave travels radially outwards.

The quantity of greatest interest, the rate of scattering of energy in a given direction, is expressed most conveniently in terms of the scattering cross-section $\sigma(\boldsymbol{\rho},\boldsymbol{\rho}')$. On the single-scattering approximation we have, using the notation of figure 1,

$$\sigma(\boldsymbol{\rho},\boldsymbol{\rho}') = \frac{r^2\overline{\phi_1\phi_1^*}}{V\phi_0\phi_0^*} \quad \text{(where the asterisk denotes complex conjugate)},$$

$$= \frac{\kappa^4 r^2}{(4\pi)^2 V}\iint \exp\left[\mathrm{i}\kappa(s'-s'')+\mathrm{i}\boldsymbol{\kappa}.\mathbf{s}\right](s's'')^{-1}\overline{f(\mathbf{r}')f(\mathbf{r}'')}\,\mathrm{d}\mathbf{r}'\,\mathrm{d}\mathbf{r}''$$

$$= \frac{\kappa^4\eta^2 r^2}{(4\pi)^2 V}\iint \exp\left[\mathrm{i}\kappa(s'-s'')+\mathrm{i}\boldsymbol{\kappa}.\mathbf{s}\right](s's'')^{-1}B(\mathbf{r}')C(\mathbf{s})\,\mathrm{d}\mathbf{r}'\,\mathrm{d}\mathbf{s}.$$

We now wish to approximate to $s'-s''$ by $-\mathbf{s}.\boldsymbol{\rho}'$. Since $B \doteqdot 0$ if $r' \gg M$, and $C \doteqdot 0$ if $s \gg L$, then for points which make a significant contribution to the integral the error in the phase

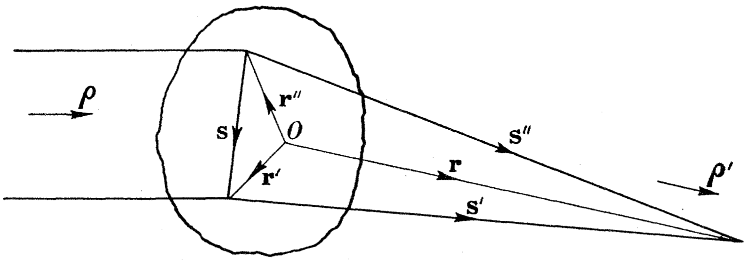

FIGURE 1. Incident wave vector $\boldsymbol{\kappa} = \kappa\boldsymbol{\rho}$. Scattered wave vector $\kappa\boldsymbol{\rho}'$.

$\kappa(s'-s'')$ will be small if $\kappa LM \ll r$. If also $M \ll r$, so that we can replace $s's''$ in the denominator by r^2, the scattering cross-section can be put into its usual form:

$$\sigma(\boldsymbol{\rho},\boldsymbol{\rho}') = \frac{\kappa^4\eta^2}{(4\pi)^2 V}\int B(\mathbf{r}')\,\mathrm{d}\mathbf{r}'\int \mathrm{e}^{-\mathrm{i}\boldsymbol{\chi}.\mathbf{s}}C(\mathbf{s})\,\mathrm{d}\mathbf{s}$$

$$= \tfrac{1}{2}\pi\kappa^4\eta^2\Phi(\boldsymbol{\chi}).$$

The scattering cross-section for all the cases referred to in § 3 can be summarized in the formula

$$\sigma(\boldsymbol{\rho},\boldsymbol{\rho}') = \tfrac{1}{2}\pi\kappa^4\eta^2G^2\Phi(\boldsymbol{\chi})$$

$$= \tfrac{1}{2}\pi\kappa^4\eta^2G^2\Phi(\kappa\boldsymbol{\rho}'-\kappa\boldsymbol{\rho}), \tag{6}$$

provided $r \gg M$, and $r \gg \kappa LM$. Here G is (*a*) unity, (*b*) and (*c*) $\cos\theta$, (*d*) $\sin\gamma$, where θ is the angle between $\boldsymbol{\rho}'$ and $\boldsymbol{\rho}$, γ the angle between $\boldsymbol{\rho}'$ and the direction of polarization of the incident vector wave in case (*d*).

When the irregularities are isotropic, Φ depends only on $\chi = 2\kappa\sin\frac{1}{2}\theta$. In cases (*a*), (*b*) and (*d*) we can then express $\eta^2\Phi(\boldsymbol{\chi})$ in terms of $\Gamma(2\kappa\sin\frac{1}{2}\theta)$, the spectrum function of the irregularities integrated over all directions of wave vector (for a full discussion of the spectrum of a convected scalar in the presence of turbulence, see Batchelor 1959), and then

$$\sigma(\boldsymbol{\rho},\boldsymbol{\rho}') = \frac{\kappa^2G^2}{32\sin^2\frac{1}{2}\theta}\Gamma(2\kappa\sin\tfrac{1}{2}\theta).$$

In case (*c*), $\eta^2\Phi(\boldsymbol{\chi}) = (2\kappa c_0)^{-2}\kappa_i\kappa_j\Phi_{ij}(\boldsymbol{\chi})$, where Φ_{ij} is the energy spectrum tensor of the turbulence, and with isotropy this can be expressed in terms of $E(2\kappa\sin\frac{1}{2}\theta)$, the usual energy spectrum function. Then

$$\sigma(\boldsymbol{\rho},\boldsymbol{\rho}') = \frac{\kappa^2\cos^2\theta}{8c_0^2\tan^2\frac{1}{2}\theta}E(2\kappa\sin\tfrac{1}{2}\theta).$$

If the smallest scale of the irregularities is much larger than the wavelength, $\Phi(\kappa\boldsymbol{\rho}')$ is extremely small, and the scattered radiation is confined to small angles of deviation. In these circumstances, the angle γ in case (*d*) is practically 90° for almost all the scattered energy, and the factor G can be put equal to unity. Hence the scattering cross-section ceases to depend on the direction of polarization, and it is only under these conditions that multiple scattering of a vector wave can strictly be described in terms of a single intensity distribution, as in equation (1).

For more detail on the calculations and results, see Booker & Gordon (1950), Villars & Weisskopf (1955), Kraichnan (1953), Lighthill (1953), Mintzer (1953*a*), Batchelor (1955, 1957).

5. Single scattering by an infinite slab

The limitation that the results given in the last section apply only at distances which are large compared with both M and κLM can sometimes be overcome by dividing the scatterer into subregions, so that those results can be applied to the energy scattered from each, and summing all the contributions. Average energy fluxes in a given direction can be added algebraically, because the contributions from the different subregions are uncorrelated. In this situation, energy will be received from an appreciable solid angle of directions, instead of from apparently a point source. (The same idea is applied in a more elaborate way in the treatment of multiple scattering in § 8.)

The case of a plane wave incident normally or obliquely on an infinite slab, containing irregularities which are stationary random functions of the co-ordinates parallel to the slab, can be treated in this way, provided that the slab is thin enough, and that a negligible proportion of the scattered energy travels nearly parallel to the slab (so that single scattering is a valid description of the process). Let us use axes (x,y,z), with the slab perpendicular to the z-axis, and of effective thickness M, and write the incident wave vector $\kappa\boldsymbol{\rho} = (\kappa\lambda,\kappa\mu,\kappa\nu)$ $(\nu > 0)$. Then we should say that, for z sufficiently large and positive, the intensity of scattered energy in direction $\boldsymbol{\rho}' = (\lambda',\mu',\nu')$ is a fraction

$$(1/\nu')\,M\sigma(\boldsymbol{\rho},\boldsymbol{\rho}') = (1/\nu')\,\tfrac{1}{2}\pi\kappa^4\eta^2MG^2\Phi(\kappa\boldsymbol{\rho}'-\kappa\boldsymbol{\rho}) \qquad (7)$$

of the energy flux (net flux as defined in § 2 (ii)) in the incident wave.

However, another way of looking at this problem has been developed by Booker, Ratcliffe & Shinn (1950), Hewish (1951), Fejer (1953) and others, and is quite illuminating. (Their work refers to electromagnetic waves, but the results will here be given for sound waves. There is no difference in principle.) If we know the complex wave function $\phi(x,y,z)$ over a plane z = constant (on the positive side of the slab), we can make a Fourier analysis over the plane, and a component $\exp[\mathrm{i}\kappa(\lambda x+\mu y)]$ corresponds to a wave travelling in direction $\{\lambda,\mu,\surd(1-\lambda^2-\mu^2)\}$. Fourier components for which the square root is imaginary are unimportant—they represent a near field carrying no power, and they decay exponentially with z.

Then we can define precisely the angular distribution of intensity (or angular power spectrum), from the average power carried by these Fourier components. From the results obtained in the references, it can be seen that the intensity in direction (λ, μ, ν) is

$$\tfrac{1}{2}Y\frac{\kappa^2\nu}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left[-\mathrm{i}\kappa(\lambda\xi+\mu\eta)\right]\overline{\phi^*(x,y,z)\,\phi(x+\xi,y+\eta,z)}\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{8}$$

where Y is a constant. This intensity is independent of z.

To apply this to the problem of a slab $(-a<z<0)$ containing irregularities which are statistically stationary in the x and y directions, we write the covariance of $f(\mathbf{r}) = f(x,y,z)$

$$\begin{aligned}\overline{f(\mathbf{r}-\tfrac{1}{2}\mathbf{s})f(\mathbf{r}+\tfrac{1}{2}\mathbf{s})} &= \eta^2R(\mathbf{s},z)\\ &= \eta^2\int\Phi(\boldsymbol{\kappa},z)\,\mathrm{e}^{\mathrm{i}\boldsymbol{\kappa}\cdot\mathbf{s}}\,\mathrm{d}\boldsymbol{\kappa}.\end{aligned}$$

With a plane incident wave in direction $\boldsymbol{\rho}$, we obtain the scattered wave for $z>0$ on the single-scattering approximation. Then, by a calculation similar to that of Ellison (1951), we find that the intensity in direction $\boldsymbol{\rho}'$ is a fraction

$$(1/\nu')\tfrac{1}{2}\pi\kappa^4\eta^2G^2\int_{-a}^{0}\Phi(\kappa\boldsymbol{\rho}'-\kappa\boldsymbol{\rho},z)\,\mathrm{d}z$$

of the energy flux in the incident wave. If the correlation function can be expressed in the form (2), where $B(\mathbf{r})$ is now a function of z alone, and $M=\int_{-a}^{0}B(z)\,\mathrm{d}z$, we obtain the result (7), which is thus shown to hold at any distance from the slab, the only approximation being that of single scattering.

The relation between the angular power spectrum and the transverse correlation is also useful in that it can be interpreted as a relation between the degree of coherence of the wave, at right angles to the direction of propagation, and the amount of angular spread in the direction of propagation. The narrower the cone of directions to which the energy is confined, the greater the lateral distance over which the wave is coherent. If this distance is called L_n (a sort of length scale of the irregularities in the wave front) then the angular spread in direction is of order $(\kappa L_n)^{-1}$.

6. Range of validity of the single-scattering approximation

It can clearly be shown that, if the scattering volume is small enough, an iteration solution of (4) will converge, and the first two terms will give a good approximation to the solution. But the upper limit which is imposed on the volume by a rigorous proof of this kind is far too small, because we have to greatly over-estimate the integrals.

It should be possible to make a more realistic estimate of the range of validity of the single-scattering approximation by calculating the twice scattered power, and requiring that this should be small compared to the singly scattered power. For on physical grounds it seems probable that, if this is satisfied, the power scattered three times will be smaller by a similar factor, and so on.

The calculations are given in outline for case (*a*) of § 3. They are more involved in the other cases, but the calculation has been done for electromagnetic waves, leading to the same results, and it seems that these should hold in general. We write

$$\phi = \phi_0 + \phi_1 + \phi_2 + \ldots,$$

with

$$\phi_0 = A\,e^{i\boldsymbol{\kappa}.\mathbf{r}},$$

$$\phi_1 = -A\frac{\kappa^2}{4\pi}\int \exp\left[i\kappa\,|\mathbf{r}-\mathbf{r}'| + i\boldsymbol{\kappa}.\mathbf{r}'\right]|\mathbf{r}-\mathbf{r}'|^{-1} f(\mathbf{r}')\,d\mathbf{r}',$$

$$\phi_2 = A\frac{\kappa^4}{16\pi^2}\iint \exp\left[i\kappa\,|\mathbf{r}-\mathbf{r}'| + i\kappa\,|\mathbf{r}'-\mathbf{r}''| + i\boldsymbol{\kappa}.\mathbf{r}''\right]|\mathbf{r}-\mathbf{r}'|^{-1}\,|\mathbf{r}'-\mathbf{r}''|^{-1} f(\mathbf{r}')f(\mathbf{r}'')\,d\mathbf{r}'\,d\mathbf{r}'',$$

and so on.

We shall simply estimate the contributions of the various terms to $\overline{\phi\phi^*}$. Thus

$$\overline{\phi_0\phi_0^*} = A^2, \quad \text{taking } A \text{ to be real;}$$

$$\overline{\phi_1\phi_1^*} = A^2\frac{\pi}{2}\frac{\kappa^4\eta^2 V}{r^2}\Phi(\boldsymbol{\chi}), \quad \text{as already calculated;}$$

$$\overline{\phi_0\phi_1^*} = 0, \quad \text{since } \overline{f(\mathbf{r})} = 0.$$

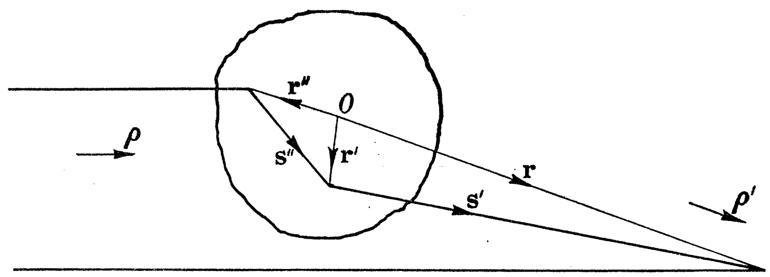

FIGURE 2

Next, still using (2) for the covariance of $f(\mathbf{r})$, and with the notation of figure 2, we have

$$\overline{\phi_0^*\phi_2} = A^2\frac{\kappa^4\eta^2}{16\pi^2}\iint \exp\left[i\kappa(s'+s'') - i\boldsymbol{\kappa}.(\mathbf{s}'+\mathbf{s}'')\right](s's'')^{-1}B(\mathbf{r}')\,C(\mathbf{s}'')\,d\mathbf{r}'\,d\mathbf{s}''$$

$$= A^2\frac{\kappa^4\eta^2}{16\pi^2}\int \exp\left[i\kappa s'' - i\boldsymbol{\kappa}.\mathbf{s}''\right](s'')^{-1}C(\mathbf{s}'')\,d\mathbf{s}''\int \exp\left[i\kappa s' - i\boldsymbol{\kappa}.\mathbf{s}'\right](s')^{-1}B(\mathbf{r}')\,d\mathbf{r}'. \qquad (9)$$

Now the second integral, if r is large enough, is approximately equal to

$$\exp\left[i\kappa r - i\boldsymbol{\kappa}.\mathbf{r}\right]r^{-1}\int \exp\left[-i\boldsymbol{\chi}.\mathbf{r}'\right]B(\mathbf{r}')\,d\mathbf{r}',$$

where, as before, $\boldsymbol{\chi} = \kappa\boldsymbol{\rho}' - \boldsymbol{\kappa}$. $B(\mathbf{r})$ gives the variation of mean-square fluctuation over the scattering volume, and cannot change appreciably in a distance of order L. Hence its Fourier transform must decrease to zero much more rapidly than $\Phi(\boldsymbol{\chi})$, as χ (and hence θ) increases. The first integral in (9) does not depend on the position $\mathbf{r}$. Thus (9) should be appreciable only for angles θ which are small compared to the angle of scattering of the bulk of the energy.

But let us calculate the total flux of energy associated with (9) over a plane perpendicular to the direction of propagation of the incident wave, on the far side of the scatterer. It is, without approximation,

$$\tfrac{1}{2}Y\mathscr{R}\int(-\mathrm{i}\kappa^{-2})\,\boldsymbol{\kappa}.\overline{(\phi_2^*\,\nabla\phi_0+\phi_0^*\,\nabla\phi_2)}\,\mathrm{d}S \quad \text{(integrating over the plane } \boldsymbol{\kappa}.\mathbf{r}=\text{constant)}$$

$$=\tfrac{1}{2}YA^2\frac{\kappa^4\eta^2}{16\pi^2}\mathscr{R}\int\exp\left[\mathrm{i}\kappa s-\mathrm{i}\boldsymbol{\kappa}.\mathbf{s}\right]s^{-1}C(\mathbf{s})\,\mathrm{d}\mathbf{s}\iint\{1+\boldsymbol{\kappa}.\mathbf{s}'(\kappa s'+\mathrm{i})\,(\kappa s')^{-2}\}$$

$$\times\exp\left[\mathrm{i}\kappa s'-\mathrm{i}\boldsymbol{\kappa}.\mathbf{s}'\right](s')^{-1}B(\mathbf{r}')\,\mathrm{d}\mathbf{r}'\,\mathrm{d}S$$

$$=YA^2\frac{\kappa^3\eta^2 V}{8\pi}\mathscr{R}\mathrm{i}\int\exp\left[\mathrm{i}\kappa s-\mathrm{i}\boldsymbol{\kappa}.\mathbf{s}\right]s^{-1}C(\mathbf{s})\,\mathrm{d}\mathbf{s}.$$

This is equal to minus the total flux of energy due to the singly scattered wave, over all directions away from the scatterer, which means, as Lighthill (1953) pointed out, that the energy lost by scattering from the main beam can be represented as a flux of energy towards the scattering volume, within the shadow region, due to the interaction of the unscattered wave with the twice scattered wave.

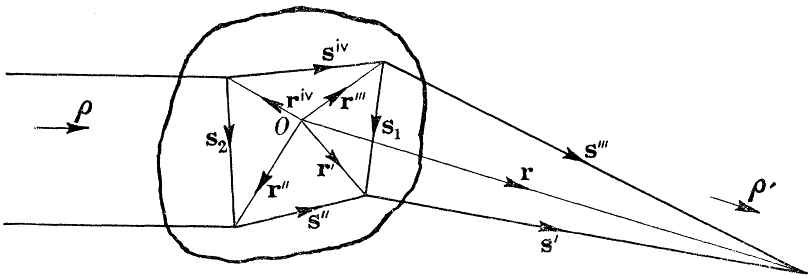

FIGURE 3

Note that the existence of a shadow is not incompatible with weak scattering, since the latter is due not to the small size of the scattering region but to the small degree of irregularity of the medium.

The next contribution to $\overline{\phi\phi^*}$ which is to be estimated is

$$\overline{\phi_2\phi_2^*}=A^2\frac{\kappa^8}{(4\pi)^4}\iiiint\exp\left[\mathrm{i}\kappa(s'+s''-s'''-s^{\mathrm{iv}})+\mathrm{i}\boldsymbol{\kappa}.\mathbf{s}_2\right]$$

$$\times(s's''s'''s^{\mathrm{iv}})^{-1}\overline{f(\mathbf{r}')f(\mathbf{r}'')f(\mathbf{r}''')f(\mathbf{r}^{\mathrm{iv}})}\,\mathrm{d}\mathbf{r}'\,\mathrm{d}\mathbf{r}''\,\mathrm{d}\mathbf{r}'''\,\mathrm{d}\mathbf{r}^{\mathrm{iv}}, \qquad (10)$$

with notation as in figure 3.

First, s' and s''' can be approximated as in the calculation in § 4, provided $r \gg M$, $r \gg \kappa LM$. The four-point mean product must be simplified in some way; an assumption which has been frequently used for random functions associated with turbulence, and which ought to give a reasonable estimate of the integral, is that the four-point mean product is related to the two-point mean products which can be formed with the same four points, according to the relation which holds for a normal joint probability distribution of the values at the points. That is,

$$\overline{f(\mathbf{r}')f(\mathbf{r}'')f(\mathbf{r}''')f(\mathbf{r}^{\mathrm{iv}})}=\overline{f(\mathbf{r}')f(\mathbf{r}'')}\,\overline{f(\mathbf{r}''')f(\mathbf{r}^{\mathrm{iv}})}+\overline{f(\mathbf{r}')f(\mathbf{r}''')}\,\overline{f(\mathbf{r}'')f(\mathbf{r}^{\mathrm{iv}})}+\overline{f(\mathbf{r}')f(\mathbf{r}^{\mathrm{iv}})}\,\overline{f(\mathbf{r}'')f(\mathbf{r}''')}.$$

The integral (10) is then written as the sum of three parts:

$$\overline{\phi_2\phi_2^*}=I_1+I_2+I_3.$$

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I_1 is equal to

$$\left| A\frac{\kappa^4\eta^2}{16\pi^2 r}\int B(\mathbf{r}')\exp\left[-\mathrm{i}\boldsymbol{\chi}.\mathbf{r}'\right]\mathrm{d}\mathbf{r}'\int \exp\left[\mathrm{i}\kappa s''-\mathrm{i}\boldsymbol{\kappa}.\mathbf{s}''\right](s'')^{-1}C(\mathbf{s}'')\,\mathrm{d}\mathbf{s}''\right|^2$$
$$= A^{-2}\,\overline{\phi_0^*\phi_2}\;\overline{\phi_0\phi_2^*}.$$

It is therefore negligible except within the shadow region. And an estimate of the contribution from I_1 to the total energy flux across a plane $\boldsymbol{\kappa}.\mathbf{r}$ = constant shows that this is of the order of $\kappa^2 LM\eta^2$ times that from $\overline{\phi_0^*\phi_2}$.

For simplicity, let the function $B(\mathbf{r})$ be spherically symmetrical, though the result is the same if it is not. Then I_2 can be expressed as

$$KA^2M\frac{\kappa^8\eta^4V}{64\pi^3r^2}\int\int \exp\left[-\mathrm{i}\kappa\boldsymbol{\rho}'.\mathbf{s}_1+\mathrm{i}\boldsymbol{\kappa}.\mathbf{s}_2\right](\kappa s)^{-1}\sin\kappa s\,C(\mathbf{s}_1)\,C(\mathbf{s}_2)\,\mathrm{d}\mathbf{s}_1\mathrm{d}\mathbf{s}_2, \tag{11}$$

where K is a constant of order unity (which depends on the form of $B(\mathbf{r})$), and $s = |\mathbf{s}_1+\mathbf{s}_2|$. If we now write C in terms of its Fourier transform Φ, the integral (11) becomes

$$I_2 = KA^2MV\frac{\pi^2\kappa^6\eta^4}{4r^2}\int \Phi(\boldsymbol{\kappa}'-\boldsymbol{\kappa})\,\Phi(\boldsymbol{\kappa}'-\kappa\boldsymbol{\rho}')\,\mathrm{d}S_{\kappa'}, \tag{12}$$

integrated over a sphere in $\boldsymbol{\kappa}'$-space, of radius κ.

Now if $|\boldsymbol{\kappa}-\kappa\boldsymbol{\rho}'| = \chi$ is large compared to L^{-1}, and if the dominant contribution to the integral of Φ over all wave numbers comes from wave numbers of order L^{-1}, which will be so in most cases, then (12) can be approximated by considering only the two areas on the sphere within L^{-1} of the points $\boldsymbol{\kappa}$ and $\kappa\boldsymbol{\rho}'$. In each of these areas, the smaller of the factors in the integrand may be replaced by its value at the centre of the area. Hence, since $\Phi(-\boldsymbol{\chi}) = \Phi(\boldsymbol{\chi})$,

$$I_2 \doteqdot KA^2MV\frac{\pi^2\kappa^6\eta^4}{2r^2}\Phi(\boldsymbol{\chi})\int \Phi(\boldsymbol{\kappa}'-\boldsymbol{\kappa})\,\mathrm{d}S_{\kappa'}$$
$$\doteqdot \text{constant}\times\kappa^2LM\eta^2\,\overline{\phi_1\phi_1^*}.$$

If χ is not large compared with L^{-1} the two areas cannot be separated in this way, but a reasonable estimate shows that I_2 should still bear the same ratio to $\overline{\phi_1\phi_1^*}$.

A similar, but more involved, calculation shows that I_3 is not of larger order than I_2, and in fact, except very close to the direction of the incident wave, is considerably smaller than I_2.

Finally, a rather rough estimate of $\overline{\phi_1^*\phi_2}$ indicates that its ratio to $\overline{\phi_1\phi_1^*}$ is smaller than $\kappa\eta L$, and much smaller than $(\kappa^2LM\eta^2)^{\frac{1}{2}}$.

From these results, the conclusion may reasonably be drawn that, if $\kappa^2LM\eta^2 \ll 1$, multiple scattering is negligible, except in directions making angles of order $\kappa^{-1}M^{-1}$ with the direction of incidence. Within this narrow cone the flux of energy in the main beam is weakened by an amount equal to the total rate at which energy is scattered.

Mintzer (1953 *b*) estimated the range of validity of single scattering, for somewhat different conditions, by requiring that the mean of ϕ_2 should be small compared to the amplitude of the incident wave. But this does not seem to give any guarantee that the power carried by ϕ_2 is negligible, since ϕ_1 has zero mean, and still carries the singly scattered power. However, the condition which he obtained is effectively the same as that which has just been found.

The average distribution of scattered energy, then, from a bounded region of irregularities satisfying $\kappa^2LM\eta^2 \ll 1$, is given by the scattering cross-section (6) in the same way as for a volume element of a uniform scattering atmosphere. But since the volume is of finite size, the radiation received at any point is spread in direction over an angle equal to that subtended by the volume, so that the scattering cross-section is really an asymptotic result for large distances. Also, the shadow cast by the volume is subject to broadening by diffraction, though it is always narrow compared to a cone containing a substantial portion of the scattered power.

When the condition that has just been derived is not satisfied, so that multiple scattering occurs, and we wish to describe the distribution of average intensity by the equation of transfer (1), we must divide a large region into volume elements, and apply to each the results of the single-scattering theory. In order that it may be possible to choose the size, M, of the elements to satisfy the condition for single scattering, $\kappa^2 LM\eta^2 \ll 1$, and also $L \ll M$, and $\kappa^{-1} \ll M$, it is sufficient that η and $\kappa\eta L$ should both be small.

The function $B(\mathbf{r})$ must now be unity over the volume element, and zero elsewhere. Hence the covariance of $f(\mathbf{r})$ cannot have the form (2) for points close to the boundary, but this will have negligible effect since $L \ll M$. Also, the sharp boundary would produce diffraction fringes around the shadow if the element were in isolation, but these can be ignored because it is in fact part of a larger volume.

7. The meaning of average intensity in a given direction in a randomly inhomogeneous medium

The definitions in § 2 depend on the concept of intensity of radiation in a given direction, which is really more appropriate to a ray treatment of scattering than to the wave treatment used in the single-scattering theory. So it is desirable to see how precisely the intensity can be defined in terms of the wave treatment.

As was stated in § 5, if the complex wave function is given over a plane $z = z_0$ as a stationary random function of x and y, for a wave travelling into homogeneous medium in $z > z_0$, then a Fourier resolution of the wave leads to a precise definition of the angular distribution of intensity in terms of the correlation function over the plane. If waves travelling out of, as well as into, the region are involved, then correlations of the wave function and its normal derivative are required, to separate the intensities in opposite directions.

But now we can extend this to define the intensity of a wave in an inhomogeneous medium, if all the statistical properties of the wave are independent of x and y. We suppose that the medium is made homogeneous over an arbitrarily thin slab containing the plane $z = z_0$. This will make negligible difference to the wave. And the values of the wave function and its normal derivative, over the plane, define an angular distribution of intensity. This definition is probably not completely consistent with values that would be obtained directly for the average flux of energy across surface elements in the inhomogeneous medium, because the relation between amplitude and energy flux depends on the local properties of the medium. But such discrepancies should not be of greater order than the degree of irregularity of the medium.

This can be generalized for waves which are not harmonic but stationary random functions of time, to give the distribution of intensity over all frequencies as well as directions, at any z, in terms of the values at all x, y and t. Alternatively, if the statistical properties are constant in space but not in time, the intensity distribution in this same sense can be obtained at any instant, in terms of the values of the wave function and its time derivative at all points of space.

So we can give definitions of intensity, in a randomly inhomogeneous medium, which are reasonably precise if the degree of irregularity is small, and which allow the intensity to be a function either of one space co-ordinate or of time. But if we wish to define an intensity which can be a function of more than one co-ordinate, the definition must be less precise. Suppose that the wave motion is harmonic in time, and that its statistical properties vary in all directions, but by a negligible amount over distances much less than R, where $\kappa R \gg 1$. Then, to define the angular spectrum of intensity, at a point, it seems most natural to take the three-dimensional spectrum of the complex wave function, supposed cut off outside some region which is centred at the point and of size $D \ll R$. The spectrum will be spread over a range of wave-number magnitudes, about κ, with an average deviation of order D^{-1}, and so the angular spectrum is obtained by integrating with respect to wave-number magnitude for each direction. Alternatively, we could use the two-dimensional spectrum over a plane through the point, with the function cut off in the same way.

With either approach, the angular spectrum will be blurred over angles of order $(\kappa D)^{-1}$ or DR^{-1}, whichever is greater, and it can be seen that for the highest precision D should be $(R/\kappa)^{\frac{1}{2}}$, leading to blurring over angles of order $(\kappa R)^{-\frac{1}{2}}$. If the equation of transfer is to be used in such a situation, it would seem that the blurring should be small compared to $(\kappa L)^{-1}$, which is a typical angle of scattering for the bulk of the energy. That is, $L(\kappa/R)^{\frac{1}{2}} \ll 1$. Here R could also be defined as a distance in which an appreciable fraction of the energy is scattered from a beam.

A similar approach can be used if the statistical properties vary in time too, and the result is the same.

In the next section, by a more detailed argument, the condition for the validity of equation (1) is found to be $\kappa L^2 R^{-1} \ll 1$.

8. Application of the equation of transfer to scattering by random irregularities

We have seen that a large region of random irregularities for which η and $\kappa\eta L$ are small can be divided into volume elements of such size that the average angular distribution of intensity scattered by each in the absence of the others, from a plane wave, is given by the cross-section for single scattering, $\sigma(\boldsymbol{\rho}, \boldsymbol{\rho}')$, which depends only on the statistical properties of the medium. But in a problem of multiple scattering, the radiation incident on any element is the result of scattering by all other elements, and we aim to describe the scattering of this radiation using the same parameter $\sigma(\boldsymbol{\rho}, \boldsymbol{\rho}')$. The waves coming from different elements can be considered separately, because they are uncorrelated; thus the question is: does the scattering by one volume element of waves arising from scattering by another element lead to the same average distribution of intensity as the scattering of a plane wave?

To consider the matter intuitively, it would seem (as suggested by Lighthill 1953) that if a wave incident on a scattering element is approximately plane, over a distance several times the size of the irregularities, then the average distribution of scattered intensity should be the same as if the incident wave were exactly plane. For since all correlation in the medium is lost over a distance a few times the size of the irregularities, lack of coherence of the incident wave over distances rather greater than this should have no effect.

Alternatively, a wave of wave number κ which is approximately plane over distances of order D, but not over much larger distances, can be represented by a set of plane waves having directions lying within a cone of angle less than $(\kappa D)^{-1}$. Since the angular power distribution resulting from the scattering of each of these will not normally show any appreciable variation over angles small compared with $(\kappa L)^{-1}$, the slight spread of the incident wave will be negligible if $D \gg L$.

Hence it may be expected that the answer to the question is, yes, provided that the distance, r_1, between the two elements is great enough. If we estimate the angular spread of the waves incident on the second element (arising from scattering by the first) as M/r_1, then the condition is $r_1 \gg \kappa ML$, which is the same as the condition (after equation (6)) for

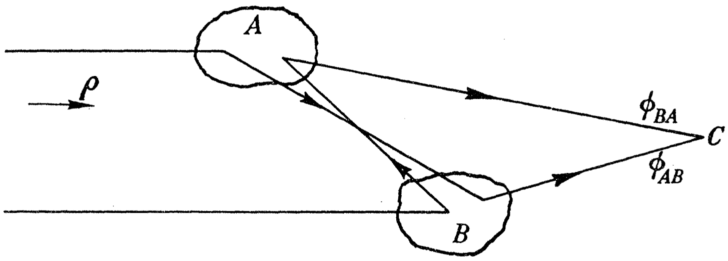

FIGURE 4

the average power reaching the second element to be that given by the cross-section for single scattering.

Supposing this to be correct, a sufficient condition for the success of the projected method of treating multiple scattering is that almost all of the radiant energy incident on each volume element should have come from scattering at distances which are large compared with κML.

However, let us first see whether the condition is necessary, by calculating the energy distribution resulting from two scatterings. We seek the average power received at C due to scattering of a plane wave by the volume elements A and B (figure 4). If it is found that the angular distribution of intensity of the wave scattered at B, after being scattered from a plane wave at A, is the same as would be given by two applications of the scattering cross-section, it follows that the scattering at B is the same, on the average, whether the incident wave is a plane wave or the result of the scattering of a plane wave at A. But it then seems a reasonable extrapolation to say that it will be the same if the incident wave is the result of the scattering of any wave at A.

We shall denote each scattered wave by a set of subscripts A and B, indicating in which elements the successive scatterings occur. Thus ϕ_{AAB} is a wave scattered twice in A and then once in B. ϕ_0 is the incident plane wave. Now, from the calculations of § 6, the only significant contributions to the power scattered at A (if the conditions for single scattering are satisfied) are $\phi_A \phi_A^*$ and $\phi_0^* \phi_{AA}$. Hence, for the power scattered by the two volume elements, we may reasonably consider only interactions which do not involve more than two scatterings in each element. Further, because the two elements are separate, each mean product of $f(\mathbf{r})$ can be factorized into two parts, each involving points in only one element;

hence interactions, such as $\overline{\phi_A^* \phi_{BB}}$, which involve only one scattering in one of the elements contribute nothing to the average power. The interactions to be considered are:

$$\phi_{AB}\phi_{AB}^*, \quad \phi_{AA}\phi_{BB}^* \qquad \phi_{AB}\phi_{BA}^* \qquad \phi_{BA}\phi_{BA}^*$$
$$\phi_A^*\phi_{ABB}, \quad \phi_B^*\phi_{AAB} \qquad \phi_A^*\phi_{BAB}, \quad \phi_B^*\phi_{ABA} \qquad \phi_A^*\phi_{BBA}, \quad \phi_B^*\phi_{BAA}$$
$$\phi_0^*\phi_{AABB} \qquad \phi_0^*\phi_{ABAB}, \text{ etc.} \qquad \phi_0^*\phi_{BBAA}.$$

The terms on the right are obtained from those on the left merely by interchanging A and B, and need not be considered separately.

The terms in the middle are interactions which involve waves travelling from A to B and from B to A. It can be seen, from the fact that the interacting waves at C are travelling in different directions, and it can also be checked by calculation, that the term $\overline{\phi_{AB}\phi_{BA}^*}$ is not responsible for any scattered intensity (except for special configurations, and the total effect of the term for all these configurations is negligible). The other terms of this type can also be shown to be negligible. It is essential that this should be so if we are to consider the radiant energy as scattered by different elements in turn, and to treat the radiation from different elements as uncorrelated.

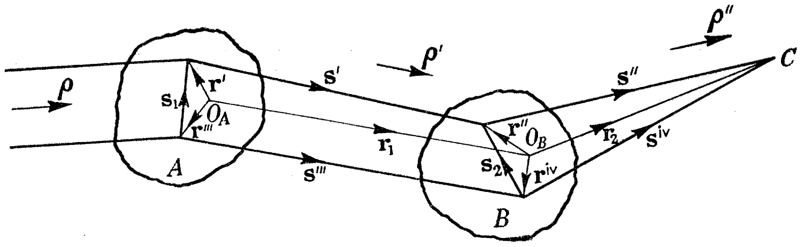

FIGURE 5

The term $\overline{\phi_{AB}\phi_{AB}^*}$ gives the power scattered at A then at B. The other four terms on the left give the loss of power from the radiation incident on A or B or both, in case B is in A's shadow or C in B's shadow. They can best be interpreted by taking them together with the incident and singly scattered power. Thus

$$\mathscr{R}(\phi_0\phi_0^* + 2\phi_0^*\phi_{AA} + 2\phi_0^*\phi_{BB} + 2\phi_0^*\phi_{AABB} + 2\phi_{AA}\phi_{BB}^*)$$

gives the power in the main beam at C minus any loss it has suffered if C is in the shadow of A or B or both (A in front of B). And

$$\mathscr{R}(\phi_A\phi_A^* + 2\phi_A^*\phi_{ABB} + \phi_B\phi_B^* + 2\phi_B^*\phi_{AAB})$$

gives the singly scattered power at C from A and B minus any loss if B is in the shadow of A, or if B is shading C from A.

We have now to calculate the average rate at which C receives energy scattered by A then by B. Per unit area it is

$$\tfrac{1}{2}Y\overline{\phi_{AB}\phi_{AB}^*} = \tfrac{1}{2}\,YA^2\frac{\kappa^8}{(4\pi)^4}\int_A\int_A\int_B\int_B \exp\left[i\kappa(s'+s''-s'''-s^{iv}) + i\boldsymbol{\kappa}.\mathbf{s}_1\right]$$
$$\times (s's''s'''s^{iv})^{-1}\overline{f_A(\mathbf{r}')f_B(\mathbf{r}'')f_A(\mathbf{r}''')f_B(\mathbf{r}^{iv})}\,\mathrm{d}\mathbf{r}'\,\mathrm{d}\mathbf{r}''\,\mathrm{d}\mathbf{r}'''\,\mathrm{d}\mathbf{r}^{iv}, \quad (13)$$

with the notation of figure 5. As already mentioned, the four-point mean product can be factorized as $\overline{f_A(\mathbf{r}')f_A(\mathbf{r}''')}\;\overline{f_B(\mathbf{r}'')f_B(\mathbf{r}^{iv})}$, which is equal to $\eta^4 B_A(\mathbf{r}')\,B_B(\mathbf{r}'')\,C(\mathbf{s}_1)\,C(\mathbf{s}_2)$, from (2), with B_A, B_B equal to unity over the respective volume elements, and zero outside them.

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As in the calculation of singly scattered power, we approximate to the phase

$$\kappa(s'+s''-s'''-s^{\text{iv}});$$

in this case, however, we put $\boldsymbol{\rho}', \boldsymbol{\rho}''$ for the unit vectors in directions $\mathbf{s}', \mathbf{s}''$. They thus depend on the variables of integration, and the condition for the approximation to be valid becomes weaker.

$$\kappa(s'+s''-s'''-s^{\text{iv}}) \doteqdot \kappa\boldsymbol{\rho}'.(\mathbf{s}_2-\mathbf{s}_1)-\kappa\boldsymbol{\rho}''.\mathbf{s}_2,$$

with an error which is small, for significant contributions to the integral (13), if s', s'' are large compared with κL^2. In the denominator of the integrand, s''' and s^{iv} can be replaced by s' and s'' if they are large compared with L.

So, if the least distances between A and B, and between B and C, are large compared with both L and κL^2,

$$\begin{aligned}\overline{\phi_{AB}\phi_{AB}^*} &= \phi_0\phi_0^*\frac{\kappa^8\eta^4}{(4\pi)^4}\int_A\int_A\int_B\int_B \exp[\mathrm{i}\kappa(\boldsymbol{\rho}'-\boldsymbol{\rho}'').\mathbf{s}_2+\mathrm{i}\kappa(\boldsymbol{\rho}-\boldsymbol{\rho}').\mathbf{s}_1] \\ &\qquad \times(s's'')^{-2}C(\mathbf{s}_1)\,C(\mathbf{s}_2)\,\mathrm{d}\mathbf{r}'\,\mathrm{d}\mathbf{r}''\,\mathrm{d}\mathbf{s}_1\,\mathrm{d}\mathbf{s}_2 \\ &= \phi_0\phi_0^*\int_A\int_B (s's'')^{-2}\tfrac{1}{4}\pi^2\kappa^8\eta^4\Phi(\kappa\boldsymbol{\rho}-\kappa\boldsymbol{\rho}')\,\Phi(\kappa\boldsymbol{\rho}'-\kappa\boldsymbol{\rho}'')\,\mathrm{d}\mathbf{r}'\,\mathrm{d}\mathbf{r}'' \\ &= \phi_0\phi_0^*\int_A\int_B (s's'')^{-2}\sigma(\boldsymbol{\rho},\boldsymbol{\rho}')\,\sigma(\boldsymbol{\rho}',\boldsymbol{\rho}'')\,\mathrm{d}\mathbf{r}'\,\mathrm{d}\mathbf{r}''. \end{aligned} \tag{14}$$

This integral is exactly what would be obtained on the assumptions that every particle of the elements A and B scatters in accordance with the cross-section $\sigma(\boldsymbol{\rho},\boldsymbol{\rho}')$, and that only single scattering occurs in each element. In fact, although the scattering irregularities are not discrete, so that we cannot calculate the cross-section by taking a volume of size comparable with L, yet the elements of size M ($\gg L$) scatter as though they were composed of discrete parts of size L, each having the scattering cross-section σ.

Similar calculations can be carried out for the average values of the other terms, $\phi_A^*\phi_{ABB}$, $\phi_B^*\phi_{AAB}$, $\phi_0^*\phi_{AABB}$, $\phi_{AA}\phi_{BB}^*$, with the results which would be expected from the physical interpretations we have given.

We can then state that, if the separation between the volume elements A and B is large compared to L and κL^2, the angular distribution of intensity scattered from B after being scattered from a plane wave at A is just what would be given by two applications of the scattering cross-section σ. And there seems to be no reason why calculations of scattering by three or more elements in turn should not give the same results.

Hence, in accordance with the remarks made earlier in this section, and from the lack of correlation between the waves scattered by different elements, we draw the conclusion that the scattering by a volume element, of energy reaching it from scatterings at distances large compared with L and κL^2, is the same as if it were a homogeneous element in an ideal scattering atmosphere, having cross-section for scattering $\sigma(\boldsymbol{\rho},\boldsymbol{\rho}')$. So we can describe the variation of average intensity of radiation by the equation of transfer (1), if the radiation travels a distance large compared with L and κL^2 before an appreciable fraction of it is scattered. This is less restrictive than the sufficient condition obtained from the argument at the beginning of this section.

The distance in which an appreciable fraction of a beam of radiation is scattered is of the order of the reciprocal of the scattering coefficient β. But $\beta(\boldsymbol{\rho}) = \int\sigma(\boldsymbol{\rho},\boldsymbol{\rho}')\,\mathrm{d}\Omega'$ integrated

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over all directions of $\boldsymbol{\rho}'$, and this is not of greater order than $\kappa^2\eta^2L$. Then the condition is satisfied if βL and $\beta\kappa L^2$ are both small, and so if

$$\eta^2 \ll (\kappa L)^{-3} \tag{15}$$

and $$\eta \ll 1.$$

The other condition which was imposed at the end of § 6 is included in these. It is not clear, of course, just what order of smallness is required.

The following points may be noted.

The whole argument remains valid if the statistical properties of the medium (and hence the scattering cross-section) vary with position, provided that negligible change occurs over distances of order L.

For electromagnetic waves, it is only when single scattering is confined to angles rather less than 90° that the dependence of scattering cross-section on the direction of polarization becomes sufficiently weak for a single equation of transfer to be applied to multiple scattering.

Although we allowed for dissipative processes in deriving the equation of transfer (1), we have not done so in the argument starting from the wave equation. If such processes occur, and are linear, their only effect is to provide a net loss of energy in each volume element, without changing the scattering cross-section. Thus $\beta(\boldsymbol{\rho})$ is increased (by a small fraction if multiple scattering is still to be important), $p(\boldsymbol{\rho}, \boldsymbol{\rho}')$ is decreased, and equation (1) applies as before.

In the usual applications, when the scattering irregularities are associated with turbulence, they have a large range of length scales, of which L is the largest, and $\kappa L \gg 1$. Then the scattering produced by the largest irregularities will be confined to angles much less than the r.m.s. angle of scattering (§ 10), and can probably be neglected without much error. So in the condition (15) we should be able to replace L by a rather smaller value, though not one comparable with the size of the smallest irregularities.

Numerical values are not given for any practical situation, because of the difficulty of getting data of sufficient accuracy to give unambiguous results.

9. Discussion of effects which have been neglected

In the previous sections we have supposed the waves to have a purely harmonic dependence on time, and we now consider briefly the situation when the wave amplitude varies with time, but not appreciably in times Lc^{-1} and $\kappa^{-1}c^{-1}$. If this is merely a result of variations in the incident wave, then the same arguments can be used to show that equation (1), which includes a time derivative, is still valid. If, however, the variation (or fading) is a result of random variations of the irregularities with time, the energy becomes spread over a band of frequencies which is narrow at first but grows wider with each scattering.

Fading has been discussed, for single scattering, by Kraichnan (1953), Mintzer (1954c), Silverman (1955, 1957). If we know the space-time correlation coefficient of the irregularities, we can calculate a generalized cross-section for single scattering which describes the distribution of energy in wave-number space (thus giving the spread in frequency as well as in direction). Then, as was mentioned at the end of § 2, the equation of transfer can be immediately generalized to describe multiple scattering in these circumstances. This assumes that the conditions laid down in § 8 are satisfied, and it may also be necessary to restrict the size of the region. For if the frequency spread became so great that frequencies comparable with those occurring in the irregularities of the medium carried an appreciable amount of radiant energy, there would be significant transfer of energy between the radiation and the random motion of the medium, and it is doubtful whether even the generalized equation of transfer would remain valid. On the other hand, until the fractional spread in frequency becomes appreciable, the ordinary equation of transfer should hold for the intensity integrated over all frequencies.

It may be as well if we consider an objection raised by Batchelor (1957, discussion), in connexion with Lighthill's work on the multiple scattering of sound by turbulent velocities. The formation of the wave equation involves neglect of certain terms before the single-scattering approximation is made, and if an iteration procedure is tried some of the neglected terms must be included in the second iteration, leading to a different scattering behaviour. Actually the same thing is true of the other wave equations, though to a lesser extent, in the neglect of squares of the fractional irregularity, and the linearization of the equations in the case of sound in a stationary inhomogeneous medium.

Considering this particular objection first, it can be seen in a rough way why it need not be incorrect to use an iteration process without inclusion of the neglected terms. For when double scattering becomes important, it does so because it receives contributions from each pair of volume elements—that is it tends to be proportional to the square of the volume of the region, whereas the single scattering is proportional to the volume. But the terms which are neglected at the first iteration are included, at the second iteration, in an integral of single scattering type—that is with the unperturbed field substituted in these terms. Hence the contribution they make to the scattering at this stage is proportional to the volume only, and is always a negligible fraction of the singly scattered energy. The same argument should apply to further iterations.

But actually, if multiple scattering is important then such an iteration procedure does not provide a usefully convergent method of solution. The present paper does not use this procedure, but sets out to show that the radiation incident on each volume element is scattered according to the same scattering cross-section as a plane wave, even though it has come from many other elements. Admittedly in the argument a double-scattering process was considered, not as part of an iteration, however, but in order to see if there is any difference between the scattering of a plane wave and one already scattered.

Perhaps the most satisfactory answer to the objection is that the neglected terms can be included in the single-scattering approximation. The calculation can still be done, and there will be only a small change in the cross-section for scattering. If the terms now included in the equation are all linear in the wave function, then the argument of § 8 applies, and the variation of intensity is described by the equation of transfer (1) with slightly modified parameters.

In the equations for sound scattering, not all the terms are linear, so that harmonics are introduced. The most important of the non-linear terms represents the self-interaction of the sound wave, leading to steepening of the wave profile, and to the formation of shocks; hence in some circumstances the methods of this paper will break down. But if the energy content of the harmonics at any point is small, either because the region of scattering is limited in size or because of attenuation by dissipative processes (which is compatible with multiple scattering if the sound amplitude is small enough), then a small modification of scattering cross-section will again lead to a satisfactory solution.

II. Some solutions of the equation of transfer

10. The equation of transfer for plane-parallel axially symmetric problems, in a statistically uniform and isotropic medium

In the special form of the equation of transfer to which this part is largely devoted, the independent variables are reduced to one space co-ordinate, x, the time t, and the x-direction cosine, μ

$$I = I(x, t, \mu).$$

The net flux vector is in the x direction, and has magnitude

$$\pi F(x, t) = 2\pi \int_{-1}^{1} \mu I(x, t, \mu)\, \mathrm{d}\mu. \tag{16}$$

Further, the absorption coefficient, β, is a constant, and the scattering function $p(\boldsymbol{\rho}, \boldsymbol{\rho}')$ depends only on the angle Θ between directions $\boldsymbol{\rho}$ and $\boldsymbol{\rho}'$. We suppose that it can be expanded in a series of Legendre polynomials

$$p(\boldsymbol{\rho}, \boldsymbol{\rho}') = p(\cos\Theta) = \sum_{0}^{\infty} \varpi_n P_n(\cos\Theta), \tag{17}$$

where

$$\cos\Theta = \boldsymbol{\rho}.\boldsymbol{\rho}',$$

and

$$\varpi_n = (n+\tfrac{1}{2}) \int_{-1}^{1} p(\mu)\, P_n(\mu)\, \mathrm{d}\mu.$$

When radiant energy is conserved, $\varpi_0 = 1$; otherwise $\beta(1-\varpi_0)$ is the coefficient of absorption by dissipative processes alone.

Another useful quantity associated with $p(\cos\Theta)$ is the r.m.s. scattering angle, α

$$\varpi_0 \alpha^2 = \frac{1}{2} \int_{0}^{\pi} \Theta^2 \sin\Theta\, p(\cos\Theta)\, \mathrm{d}\Theta,$$

and, when α is small, the ϖ_n can be expressed to a good approximation, for n not too large, in terms of α

$$\varpi_n \doteqdot \varpi_0 (2n+1) \{1 - \tfrac{1}{4} n(n+1)\, \alpha^2\} \quad \text{for} \quad n^2\alpha^2 \ll 1. \tag{18}$$

An inequality always satisfied is

$$|\varpi_n| < (2n+1)\, \varpi_0 \quad \text{for} \quad n \neq 0.$$

The equation of transfer is now

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \mu\frac{\partial}{\partial x} + \beta\right) I(x, t, \mu) = \frac{\beta}{4\pi} \int_{-1}^{1} \mathrm{d}\mu' \int_{0}^{2\pi} I(x, t, \mu')\, p(\cos\Theta)\, \mathrm{d}\phi'$$

$$= \tfrac{1}{2}\beta \int_{-1}^{1} I(x, t, \mu')\, K(\mu, \mu')\, \mathrm{d}\mu', \tag{19}$$

where
$$\cos\Theta = \mu\mu' + (1-\mu^2)^{\frac{1}{2}}(1-\mu'^2)^{\frac{1}{2}}\cos\phi',$$

and
$$K(\mu,\mu') = \frac{1}{2\pi}\int_0^{2\pi} p(\cos\Theta)\,\mathrm{d}\phi'$$
$$= \sum_0^\infty \varpi_n P_n(\mu)\,P_n(\mu').$$

Or, if we express $I(x,t,\mu)$ as a series of Legendre polynomials in μ

$$I(x,t,\mu) = \sum_0^\infty A_n(x,t)\,P_n(\mu),$$

the A_n must satisfy the equations

$$\frac{n+1}{2n+3}\frac{\partial A_{n+1}}{\partial x} + \frac{n}{2n-1}\frac{\partial A_{n-1}}{\partial x} + \frac{1}{c}\frac{\partial A_n}{\partial t} = \beta\left(\frac{\varpi_n}{2n+1}-1\right)A_n \quad \text{for} \quad n = 0, 1, 2, \ldots. \tag{20}$$

11. Solution dependent on time but not on space co-ordinates

From the equations (20), in principle, all the A_n could be found in terms of A_0 and suitable initial conditions. In fact the initial conditions are not usually given in such a form that this is convenient.

But in one case, where the intensity is independent of x, the solution is readily obtained. We can imagine at $t = 0$ a uniform field of radiation, in an infinite scattering medium, and investigate the way in which the field becomes isotropic. Equations (20) become

$$\frac{1}{c}\frac{\partial A_n}{\partial t} = \beta\left(\frac{\varpi_n}{2n+1}-1\right)A_n.$$

Hence
$$A_n(t) = A_n(0)\exp\left\{-\beta ct\left(1-\frac{\varpi_n}{2n+1}\right)\right\},$$

or
$$I(t,\mu) = \sum_0^\infty A_n(0)\,P_n(\mu)\exp\left\{-\beta ct\left(1-\frac{\varpi_n}{2n+1}\right)\right\}.$$

If the radiation is unidirectional at $t = 0$, with net flux πF, then using the Dirac δ-function we have
$$I(0,\mu) = \tfrac{1}{2}F\delta(1-\mu)$$

and
$$A_n(0) = \tfrac{1}{4}(2n+1)\,F.$$

The solution given is valid if we use an appropriate method of summing the divergent series, or if we take out of the series a part corresponding to the unscattered radiation, $I(0,\mu)\,\mathrm{e}^{-\beta ct}$, which is responsible for the divergence.

$$I(t,\mu) = \tfrac{1}{4}F\sum_0^\infty (2n+1)\,P_n(\mu)\exp\left\{-\beta ct\left(1-\frac{\varpi_n}{2n+1}\right)\right\}$$
$$= I(0,\mu)\,\mathrm{e}^{-\beta ct} + \tfrac{1}{4}F\,\mathrm{e}^{-\beta ct}\sum_0^\infty (2n+1)\,P_n(\mu)\left(\exp\frac{\beta ct\varpi_n}{2n+1}-1\right). \tag{21}$$

The largest of the quantities $\varpi_n/(2n+1)$ is always ϖ_0, so that at long times the dominant part of the solution is the isotropic radiation field,

$$I_0(t,\mu) = \tfrac{1}{4}F\exp\left[(\varpi_0-1)\,\beta ct\right],$$

either constant or decaying with time. If the scattering is mainly forward, for the usual forms of spectrum it seems very probable that the second largest of those quantities will be $\frac{1}{3}\varpi_1$, so that the manner of approach to the isotropic field is given by

$$I_1(t,\mu) = \tfrac{3}{4}F\exp\left[\left(\tfrac{1}{3}\varpi_1 - 1\right)\beta ct\right]P_1(\mu).$$

When the r.m.s. scattering angle is small, we can make the approximation given in equation (18), and so, provided t is large enough for all terms for which the approximation is not good to be negligible ($\beta ct \gg 1$),

$$I(t,\mu) \doteqdot \tfrac{1}{4}F\exp\left[(\varpi_0 - 1)\beta ct\right]\sum_0^\infty (2n+1)P_n(\mu)\exp\left[-\tfrac{1}{4}n(n+1)\varpi_0\alpha^2\beta ct\right]. \tag{22}$$

The solution which was given by Lighthill (1953), for multiple scattering with small r.m.s. scattering angle, refers to a conservative steady-state situation with a unidirectional beam incident on a region of irregularities. But he found the angular distribution of intensity for energy which has been scattered a given number of times, and then, using a Poisson distribution for the fraction of energy scattered m times in travelling a path of length l, obtained an expression for the angular distribution of intensity for radiation which has travelled a path of length l

$$\sum_0^\infty (2n+1)P_n(\mu)\exp\{-\beta l(1-\exp[-\tfrac{1}{4}n(n+1)\alpha^2])\}.$$

Since the velocity of the radiation is constant, this is equivalent to the angular distribution of intensity for radiation which has been travelling for a time $t = lc^{-1}$ in the scattering medium, and so it is equivalent to the solution which has been obtained in this section. It appears that Lighthill's solution should be more accurate than the approximate form (22), and it can be seen that it will be exact if

$$\varpi_n = (2n+1)\exp\left[-\tfrac{1}{4}n(n+1)\alpha^2\right],$$

that is if
$$p(\cos\Theta) = \sum_0^\infty (2n+1)\exp\left[-\tfrac{1}{4}n(n+1)\alpha^2\right]P_n(\cos\Theta).$$

But it does not give much information about the angular spectrum at any point in space, for the steady-state situation, since there is not a unique path length corresponding to a point in space. The formulation used in this section gives a more readily visualized solution, and one which is exact for all r.m.s. scattering angles, and all scattering functions.

12. Steady-state problem

The next simplest problem is that of steady-state conditions, with spatial variation, but this is already much more difficult. Chandrasekhar's book (1950) is largely devoted to this and related problems, with a scattering function which is either isotropic ($p(\cos\Theta) = 1$) or linear or quadratic in $\cos\Theta$.

In conservative systems (for which $\varpi_0 = 1$), two simple integral relations can be obtained; they follow directly from the first two of (20), with $\partial A_n/\partial t = 0$.

(i)
$$2\int_{-1}^{1}\mu I(x,\mu)\,\mathrm{d}\mu = F = \text{constant}. \tag{23}$$

That is, the net flux is constant. In three-dimensional steady-state conservative problems $\operatorname{div}\mathbf{F} = 0$, as is obvious physically, and this reduces to the present relation when x is the only co-ordinate involved.

(ii) $$\frac{1}{2}\int_{-1}^{1}\mu^2 I(x,\mu)\,\mathrm{d}\mu = \tfrac{1}{4}F\{-\beta x(1-\tfrac{1}{3}\varpi_1)+Q\}, \tag{24}$$

where Q is a constant.

But the integral $\int_{-1}^{1} I(x,\mu)\,\mathrm{d}\mu$, which is proportional to the energy density, does not appear to satisfy such a relation.

Associated with these relations is a simple particular solution of the equation, obtained by putting $A_2 = A_3 = A_4 = \ldots = 0$. It is

$$I(x,\mu) = \tfrac{3}{4}F\{\mu-\beta x(1-\tfrac{1}{3}\varpi_1)+Q\}. \tag{25}$$

One reason for the difficulty of the present problem is the way in which the boundary conditions are specified. When the scattering region is a slab, $0 < x < l$, or a semi-infinite region, $0 < x < \infty$, with radiation incident normally from $x < 0$, then $I(0,\mu)$ is given for $\mu > 0$ (incident radiation), but not for $\mu < 0$. For a slab, $I(l,\mu)$ is given for $\mu < 0$, but not for $\mu > 0$; for a semi-infinite region the asymptotic behaviour at infinity must be given. In this latter case the usual asymptotic forms are: for the conservative problem, the solution (25) with $-\pi F$ equal to the net flux from infinity (which must be positive or zero); and for the problem with dissipative processes, a solution of the form

$$\mathrm{e}^{\lambda\beta x}\,g(\mu) \quad (\lambda > 0,\ g(\mu) \geqslant 0).$$

Chandrasekhar shows that, when $p(\cos\Theta)$ is a finite sum of Legendre polynomials, a generalized problem in which the radiation falls obliquely on the scattering region can be solved in terms of the solution of a set of non-linear integral equations. Numerical solutions for cases where $p(\cos\Theta)$ has only one or two terms in the Legendre expansion have been published, but when there are many terms this method does not seem to be practically useful. And if the r.m.s. scattering angle is fairly small, many terms will be required even for an approximation to $p(\cos\Theta)$.

We can try to find solutions in which the variables are separated, obtaining an ordinary integral equation for the μ variation. This has a series of eigenvalues which appears to either terminate or tend to some finite limit, and the set of eigenfunctions is probably not complete in general.

But we turn now to the case of small r.m.s. scattering angle, for which some approximate solutions can be obtained. Further comments on the general case are made in § 17 (see also the work of Ueno (e.g. 1958), and of Breitenberger (1959)).

13. Small total angular deviation of the radiation

First, we consider scattering by a slab that is thin enough, and has a small enough r.m.s. scattering angle, for the total deviation of the radiation from its original direction to be small. Making appropriate approximations, expressing $I(x,\mu)$ as a function of $\theta = \arccos\mu$ $(I(x,\mu) = I_1(x,\theta))$, and writing $p(\cos\Theta) = p_1(\Theta)$, we have

$$\left(\frac{\partial}{\partial x}+\beta\right) I_1(x,\theta) = \frac{\beta}{4\pi}\int_0^{\infty}\theta'\,\mathrm{d}\theta'\int_0^{2\pi} I_1(x,\theta')\,p_1(\Theta)\,\mathrm{d}\phi', \tag{26}$$

where $$\Theta = (\theta^2+\theta'^2-2\theta\theta'\cos\phi')^{\frac{1}{2}}.$$

This equation is similar to that for the time-dependent problem, and the solution is obtained in a similar way, but in terms of Hankel transforms. When the incident beam is unidirectional ($I_1(0,\theta) = \frac{1}{2}F\theta^{-1}\delta(\theta)$), the solution is

$$I_1(x,\theta) = I_1(0,\theta)\,e^{-\beta x} + \tfrac{1}{2}F\,e^{-\beta x}\int_0^\infty yJ_0(y\theta)\,\{e^{\beta x\psi(y)}-1\}\,dy, \tag{27}$$

where
$$\psi(y) = \frac{1}{2}\int_0^\infty \Theta p_1(\Theta)\,J_0(y\Theta)\,d\Theta.$$

The problem is treated from a different angle by Fejer (1953). He assumes a simple form for the correlation function of the irregularities responsible for the scattering, and obtains a $p_1(\Theta)$ of the form $(4/\alpha^2)\exp(-\Theta^2/\alpha^2)$. He is then able to evaluate exactly the intensities corresponding to first, second, third, scattering, and so on, and to express the solution as a series

$$I_1(x,\theta) = F\alpha^{-2}\,e^{-\beta x}\sum_0^\infty \frac{(\beta x)^n}{n.n!}\exp\left(-\frac{\theta^2}{n\alpha^2}\right). \tag{28}$$

This expression can be transformed into that in (27) with the appropriate form of $\psi(y)$, $\exp(-\frac{1}{4}\alpha^2y^2)$.

Now $\psi(y) < \varpi_0$ for all $y > 0$, and so we can obtain an asymptotic expansion of (27) for large βx. Expanding $\psi(y)$ as a power series,

$$\psi(y) = \varpi_0 - \psi_1 y^2 + \psi_2 y^4 \ldots \quad (\psi_1 \doteqdot \tfrac{1}{4}\varpi_0\alpha^2),$$

we have

$$I_1(x,\theta) \sim \frac{F}{4\psi_1\beta x}\exp\left\{\beta x(\varpi_0-1) - \frac{\theta^2}{4\psi_1\beta x}\right\}\left\{1 + \frac{\psi_2}{\psi_1^2\beta x}\,{}_1F_1\left[-2;\,1;\,\frac{\theta^2}{4\psi_1\beta x}\right] + O\left(\frac{1}{\beta^2x^2}\right)\right\}, \tag{29}$$

which shows that the distribution tends to a Gaussian form, and also gives the way in which it approaches this form.

Since we have assumed that θ^2 is small, the conditions for the asymptotic form to be valid are $1 \ll \beta x \ll (4\psi_1)^{-1}$—that is $1 \ll \beta x \ll \alpha^{-2}$.

14. Approximate partial differential equation for small scattering angle, but large total angular deviation

In the previous section, we required not only a small scattering angle, but a small angular deviation from the original direction of the radiation, and so a limited thickness of the slab. Solutions can be obtained, however, without the second restriction, provided that the variation of $I(x,t,\mu)$ with μ is not too rapid. Seeking an approximate form of equation (19), like the one obtained by Lighthill (1953) for the intensity as a function of direction and number of scatterings, we expand $I(x,t,\mu')$ as a power series in $(\mu'-\mu)$. The integral then becomes

$$\int_{-1}^{1} d\mu' \int_0^{2\pi} I(x,t,\mu')\,p(\cos\Theta)\,d\phi' = 4\pi\varpi_0\left\{1 - \tfrac{1}{2}\alpha^2\mu\frac{\partial}{\partial\mu} + \tfrac{1}{4}\alpha^2(1-\mu^2)\frac{\partial^2}{\partial\mu^2} + \ldots\right\}I(x,t,\mu).$$

The most important terms of those omitted are like $(\alpha^n/n!)\,\partial^nI/\partial\mu^n$, for $n = 4, 6, 8, \ldots$, and so the remainder after the first three terms can probably be neglected if the smallest range of μ over which appreciable variations in $I(x,t,\mu)$ occur is large compared with α. Then equation (19) is

$$\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial I}{\partial\mu} = \frac{4}{\varpi_0\beta\alpha^2}\left\{\frac{1}{c}\frac{\partial I}{\partial t} + \mu\frac{\partial I}{\partial x} + \beta(1-\varpi_0)\,I\right\}. \tag{30}$$

We can obtain checks on the validity of the approximation, first, by observing that the approximation (22) to the exact solution of the variation with time of an initially unidirectional beam satisfies (30). This equation is also satisfied, for small angles of deviation, by the approximate asymptotic form $(\varpi_0\alpha^2\beta x)^{-1}F\exp\{\beta x(\varpi_0-1)-\theta^2/(\varpi_0\alpha^2\beta x)\}$ of the solution of equation (26), which was obtained without any assumptions about the magnitude of the derivatives of $I(x,t,\mu)$.

Hence it appears that, even in these cases where the initial distribution of intensity has a δ-function behaviour with μ, it rapidly acquires a form which satisfies the assumptions made in deriving (30), and the solutions to this equation have little error after many scatterings.

It is also of interest to note that, for a particular form of scattering function, the equation of transfer (19) can be expressed exactly as a differential equation (a suggestion for which I am grateful to Dr V. Hutson). $p(\mu)$ is chosen to be a Legendre function of complex order (satisfying $(\mathrm{d}/\mathrm{d}\mu)(1-\mu^2)\,\mathrm{d}p/\mathrm{d}\mu=(4/\alpha^2)\,p$, and regular at $\mu=-1$). Then we obtain, omitting time derivatives,

$$\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial I}{\partial\mu}=\frac{4}{\varpi_0\beta\alpha^2}\left\{\mu\frac{\partial I}{\partial x}+\beta(1-\varpi_0)\,I\right\}-\frac{1}{\beta}\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial}{\partial\mu}\mu\frac{\partial I}{\partial x}. \tag{31}$$

This scattering function has a logarithmic singularity at $\mu=1$ $(\Theta=0)$, and so is not particularly realistic.

15. Solutions of the approximating partial differential equation

For the steady state, we write $\frac{1}{4}\varpi_0\alpha^2\beta x=s$, $4(1-\varpi_0)=\varpi_0\alpha^2\gamma$, $(\gamma\geqslant 0)$, and then the equation (30) is

$$\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial I}{\partial\mu}=\mu\frac{\partial I}{\partial s}+\gamma I. \tag{32}$$

The same integral relations, and the same simple solution, hold (in conservative problems, where $\gamma=0$) as for the exact equation for the steady state (§ 12)

$$\left.\begin{aligned}2\int_{-1}^{1}\mu I(s,\mu)\,\mathrm{d}\mu&=F,\\ 2\int_{-1}^{1}\mu^2 I(s,\mu)\,\mathrm{d}\mu&=(-2s+Q)\,F,\end{aligned}\right\} \tag{33}$$

where Q and F are constants. The particular solution is

$$I(s,\mu)=\tfrac{3}{4}(\mu-2s+Q)\,F. \tag{34}$$

Other solutions exist in the form

$$I(s,\mu)=\mathrm{e}^{-ks}\,G(\mu),$$

where

$$\frac{\mathrm{d}}{\mathrm{d}\mu}(1-\mu^2)\frac{\mathrm{d}G}{\mathrm{d}\mu}+(k\mu-\gamma)\,G=0. \tag{35}$$

At the singular points $\mu=\pm1$, $G(\mu)$ may be regular or have a logarithmic singularity, and there are eigenvalues of k for which one solution is regular at both points, and hence for all finite μ. The eigenvalues form an infinite set, $\pm k_n$ for $n=0,1,2,\ldots$, and the eigenfunctions are $g_n(\pm\mu)$. When $\gamma=0$, then $\pm k_0=0$, $g_0(\pm\mu)\equiv 1$, and we include in the set the function $g_c(\mu)=\frac{3}{2}\mu$, which can be obtained as the limit as $\gamma\to 0$ of $\frac{3}{2}k_0^{-1}\{g_0(\mu)-g_0(-\mu)\}$, and is also the value at $s=0$ of the particular solution of (32), $I(s,\mu)=\frac{3}{2}(\mu-2s)$.

Now the usual orthogonality relations can be obtained, and, with appropriate normalization of $g_n(\mu)$, we have

$$\int_{-1}^{1} \mu g_m(\mu)\, g_n(\mu)\, \mathrm{d}\mu = \begin{cases} 0 & \text{if} \quad m \neq n, \\ 1 & \text{if} \quad m = n > 0, \\ \tfrac{2}{3}k_0 & \text{if} \quad m = n = 0, \end{cases}$$

$$\int_{-1}^{1} \mu g_m(\mu)\, g_n(-\mu)\, \mathrm{d}\mu = 0,$$

with the following addition if $\gamma = 0$

$$\int_{-1}^{1} \mu g_c(\mu)\, g_n(\pm\mu)\, \mathrm{d}\mu = \begin{cases} 0 & \text{if} \quad n > 0, \\ 1 & \text{if} \quad n = 0, \end{cases}$$

$$\int_{-1}^{1} \mu \{g_c(\mu)\}^2\, \mathrm{d}\mu = 0.$$

We may note that, of the set of solutions to (32) corresponding to these functions, the only one which carries any net flux, when $\gamma = 0$, is $\tfrac{3}{2}(\mu - 2s) = g_c(\mu) - 3s$.

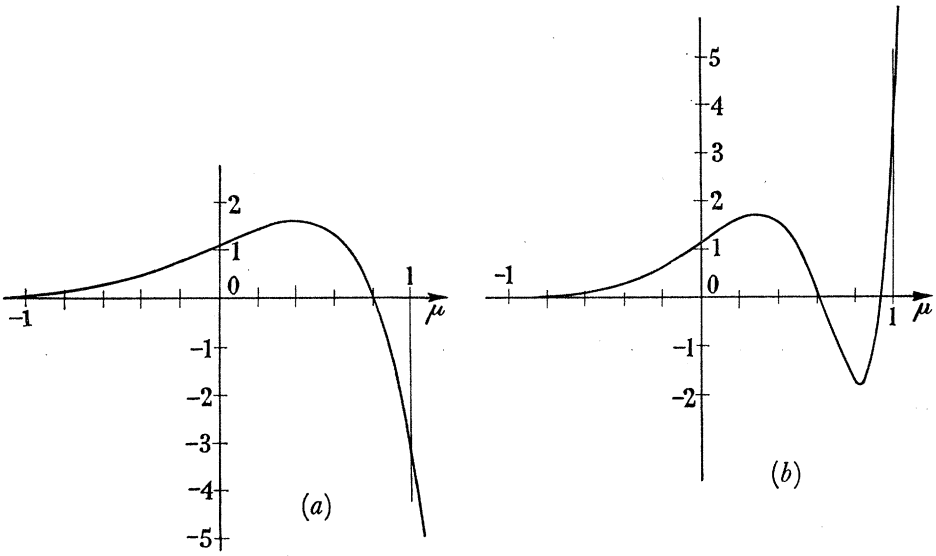

FIGURE 6. (*a*) Eigenfunction $g_1(\mu)$. $k_1 = 14{\cdot}53$; (*b*) eigenfunction $g_2(\mu)$. $k_2 = 42{\cdot}05$.

It seems likely that these functions form a complete set (see § 17), and we shall suppose that any regular solution of (32) can be expressed as a series in the functions $g_n(\mu)\, \mathrm{e}^{-k_n s}$ and $g_n(-\mu)\, \mathrm{e}^{k_n s}$, for $n = 0, 1, 2, \ldots$ (and $g_c(\mu) - 3s$ if $\gamma = 0$).

We can calculate the eigenfunctions as Legendre series, $g_n(\mu) = \sum_0^\infty a_{nm} P_m(\mu)$. From the recurrence relations which are obtained for the a_{nm}, with the condition that $a_{nm} \to 0$ as $m \to \infty$, since $g_n(\mu)$ is regular, we find that the eigenvalues are zeros of the continued fraction

$$1 - \frac{\rho_1 k^2}{1-}\,\frac{\rho_2 k^2}{1-}\,\frac{\rho_3 k^2}{1-}\ldots, \tag{36}$$

where $\rho_n = (n^2 - 1 + 2\gamma + \gamma^2 n^{-2})^{-1}\,(4n^2 - 1)^{-1}$. The first few roots have been obtained numerically, for the conservative case $\gamma = 0$; the eigenfunctions can then be readily found. The first two are shown in figure 6.

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For larger values of k, good asymptotic expansions can be derived for the solutions of (35) by extensions of the W.K.B. method, as developed by Olver (1954). In particular, when $\gamma = 0$,

$$k_n^{\frac{1}{2}} \sim (n+\tfrac{1}{2})\frac{\pi}{\Gamma}\left[1-\frac{5}{24\pi}(n+\tfrac{1}{2})^{-2}-\left\{\left(\frac{5}{24\pi}\right)^2+\frac{35}{384}\frac{\Gamma^4}{\pi^4}\right\}(n+\tfrac{1}{2})^{-4}+\dots\right]$$

for large n, where $\Gamma = \int_0^1 x^{\frac{1}{2}}(1-x^2)^{-\frac{1}{2}}\,dx = 1{\cdot}1981$. This gives even k_1 correct to four significant figures, and the other eigenvalues with greater accuracy.

When γ is not zero, but small, the first term in this expansion is the same and the others are changed by small amounts of order γ. The main difference occurs in k_0 and $g_0(\mu)$; expanding in powers of γ we find that

$$k_0^2 = 6\gamma + \tfrac{11}{5}\gamma^2 + \dots,$$

and

$$g_0(\mu) = 1 + \tfrac{1}{2}k_0\mu + \tfrac{1}{12}k_0^2(\mu^2 - \tfrac{1}{10}) + \dots.$$

Semi-infinite scattering region

We come now to the problem of constructing solutions satisfying given boundary conditions, and consider first a semi-infinite scattering region $0 < s < \infty$, with $I(s,\mu)$ taking prescribed values on $s = 0$, $0 < \mu \leqslant 1$, and having as asymptotic form for large s a given multiple of $g_0(-\mu)\,e^{k_0 s}$ (or $g_c(\mu) - 3s$). Then, if these conditions define a unique radiation field (as it seems physically they should), and if the assumption about the completeness of the set of g_n is correct, $I(s,\mu)$ can be expressed as the sum of this asymptotic form and $\sum_0^\infty v_n g_n(\mu)\,e^{-k_n s}$, where the v_n are defined uniquely by the requirement that $\sum_0^\infty v_n g_n(\mu)$ is a known function, say $\phi(\mu)$, in $0 < \mu \leqslant 1$.

We cannot take full advantage of the orthogonality, because it holds only on $(-1, 1)$, so some iteration process is required for the solution. A convenient set of equations is

$$\sum_0^\infty A_{mn} v_n = \int_0^1 \mu\{g_m(\mu) - g_m(-\mu)\}\,\phi(\mu)\,d\mu \quad \text{for} \quad m = 0, 1, 2, \dots, \tag{37}$$

where

$$A_{mn} = \int_0^1 \mu\{g_m(\mu) - g_m(-\mu)\}\,g_n(\mu)\,d\mu,$$

which can be calculated from the numerical solutions and asymptotic formulae. (When $\gamma = 0$, a different equation must be used for $m = 0$.) This form was chosen to get the greatest rate of decrease of the A_{mn} along rows and columns, as either m or n tends to infinity, but the decrease is still slow (a power $-\frac{11}{6}$ along rows and columns, and -1 along diagonal lines). However, the matrix is sufficiently close to diagonal form for iteration to be practicable, the A_{mm} being large compared with all other elements.

Solutions have been found for the following two conservative problems ($\gamma = 0$).

(*a*) A unidirectional beam, carrying energy flux πF, is incident normally on a semi-infinite region. All the radiation is ultimately reflected, so that there is no net flux.

(*b*) There is no incident radiation, but net flux πF from infinity.

Defining u_n, v_n, w_n by

$$u_0 = \tfrac{3}{2}, \quad u_n = g_n(1) \quad \text{for} \quad n > 0,$$

$$\sum_0^\infty v_n g_n(\mu) = \tfrac{3}{2}\mu \quad \text{in} \quad 0 < \mu \leqslant 1,$$

$$\sum_0^\infty w_n g_n(\mu) = \sum_1^\infty g_n(-1)\, g_n(-\mu) \quad \text{in} \quad 0 < \mu \leqslant 1,$$

we find that the solutions to those problems are given by

$$\left.\begin{aligned} &(a) \quad I(s,\mu) = \tfrac{1}{2}F \sum_0^\infty (u_n + v_n - w_n)\, g_n(\mu)\, \mathrm{e}^{-k_n s}, \\ &(b) \quad I(s,\mu) = -\tfrac{3}{4}(\mu - 2s)\, F + \tfrac{1}{2}F \sum_0^\infty v_n g_n(\mu)\, \mathrm{e}^{-k_n s}. \end{aligned}\right\} \tag{38}$$

The first few coefficients are

n	0	1	2	3	4
u_n	1·5	−3·215	4·146	−4·906	5·562
v_n	1·075	−0·198	−0·092	−0·056	−0·038
w_n	0·008	0·004	0·003	0·002	0·001

Where s is large, that is far from the surface, these solutions reduce to the forms

$$(a) \quad I(s,\mu) = 1{\cdot}28F,$$

$$(b) \quad I(s,\mu) = \tfrac{3}{4}(-\mu + 2s + 0{\cdot}72)\, F.$$

The former is a constant isotropic radiation field, and the latter has a constant anisotropic part superimposed on an isotropic part which is proportional to s. In each case, the approach of the solution to the asymptotic form is given finally by a term proportional to

$$g_1(\mu) \exp[-k_1 s] = g_1(\mu) \exp[-14{\cdot}53\, s] = g_1(\mu) \exp[-3{\cdot}63\beta\alpha^2 x],$$

which represents a much faster approach, if x is taken to correspond to ct, than in the problem with dependence on time but not on space (§ 11).

Slab of scattering medium

The problem of a slab of scattering medium, $0 < s < \frac{1}{4}\beta\alpha^2 l$, on which a unidirectional beam carrying energy flux πF is incident normally at $s = 0$, can be solved, if l is large enough, by a combination of the solutions already given

$$I(s,\mu) = \tfrac{1}{2}F\left\{y_0 + \tfrac{3}{2}z_0(\mu - 2s) + \sum_1^\infty y_n g_n(\mu) \exp[-k_n s] + \sum_1^\infty z_n g_n(-\mu) \exp[-k_n(\tfrac{1}{4}\beta\alpha^2 l - s)]\right\}, \tag{39}$$

where, provided $\exp(-\frac{1}{4}k_1\beta\alpha^2 l) = \exp(-3{\cdot}63\beta\alpha^2 l)$ is negligible, the y_n and z_n are given by

$$z_0 = \frac{u_0 + v_0 - w_0}{\frac{3}{4}\beta\alpha^2 l + 2v_0} = \frac{2{\cdot}567}{\frac{3}{4}\beta\alpha^2 l + 2{\cdot}150},$$

$$y_0 = (\tfrac{3}{4}\beta\alpha^2 l + v_0)\, z_0,$$

$$\left.\begin{aligned} z_n &= u_n + (1 - z_0)\, v_n - w_n \\ y_n &= z_0 v_n \end{aligned}\right\} \quad \text{for} \quad n = 1, 2, 3, \ldots.$$

The coefficient z_0 gives the ratio of the net flux, that is the flux emerging from the far side, to that incident on the slab, and it can be seen that the reciprocal of this is linearly related to the thickness of the slab.

The condition that $\exp(-3{\cdot}63\beta\alpha^2 l)$ should be negligible means that the disturbance produced by each surface of the slab does not extend over the whole thickness, and the radiation emerging at the far side behaves like that from a semi-infinite region.

Angular distribution of emergent radiation

The function $I(0,\mu)$ is given in $0<\mu\leqslant 1$; it is of interest to calculate it for the rest of the range of μ—that is for the emergent radiation. When s is zero and μ small the series (38) and (39) are too slowly convergent, but when μ is negative and not too small in magnitude they converge rapidly, because $g_n(\mu)$ decreases exponentially with n.

The solution to problem (*b*), for $s=0$ and $\mu<-\frac{1}{3}$, does not differ greatly from the sum of the first two terms in its series, $\frac{1}{2}F(-1{\cdot}5\mu+1{\cdot}075)$. In problem (*a*), the part of the solution containing the coefficients u_n can be transformed to a more convenient form when $s=0$, by the use of the representation of the δ-function $\delta(1-\mu)$ in terms of the full set of eigenfunctions, on the range $(-1, 1)$. We find that

$$\sum_0^\infty u_n g_n(\mu) = -\tfrac{3}{2}\mu + \sum_1^\infty g_n(-1)\, g_n(-\mu) \quad \text{in} \quad -1\leqslant\mu<1,$$

and the $g_n(-1)$ rapidly approach zero as n increases, with $g_1(-1)=0{\cdot}0385$. Then the emergent radiation is described by

$$I(0,\mu) = \tfrac{1}{2}F\left\{-\tfrac{3}{2}\mu + \sum_1^\infty g_n(-1)\, g_n(-\mu) + \sum_0^\infty (v_n - w_n)\, g_n(\mu)\right\},$$

which is not greatly different from the distribution for problem (*b*) (as would be expected).

To find the behaviour of $I(s,\mu)$ near $\mu=s=0$, we can use an approximate similarity solution of equation (32). If the terms $(\partial/\partial\mu)\,\mu^2\,\partial I/\partial\mu$ and γI are omitted, the equation can be satisfied by

$$I(s,\mu) = s^m f(\tfrac{1}{9}\mu^3 s^{-1}) = s^m f(r), \tag{40}$$

where

$$rf'' + (\tfrac{2}{3}+r)f' - mf = 0.$$

$I(0,\mu)$ is then determined by the asymptotic behaviour of $f(r)$ as $r\to\pm\infty$; a solution such that $I(0,\mu)$ is zero in $\mu>0$, and finite in $\mu\leqslant 0$, exists only if m has one of the values $\frac{1}{6}, 1\frac{1}{6}, 2\frac{1}{6}, \ldots$. If this is so, $I(0,\mu)$ is proportional to $(-\mu)^{3m}$ in $\mu\leqslant 0$. The appropriate value to use is $m=\frac{1}{6}$.

Further, it appears to be possible to satisfy exactly the full equation (32), with the condition $I(0,\mu)=0$ in $\mu>0$, by a series

$$I(s,\mu) = \sum_0^\infty s^{\frac{1}{6}(4n+1)} f_n(\tfrac{1}{9}\mu^3 s^{-1}) \quad (f_0 \equiv f),$$

leading to

$$I(0,\mu) = \sum_0^\infty b_n(-\mu)^{2n+\frac{1}{2}} \quad \text{in} \quad \mu\leqslant 0.$$

Thus the intensity of emergent radiation varies as $(-\mu)^{\frac{1}{2}}$ for μ small and negative, and the intensity, $I(s,0)$, within the medium and parallel to the surface varies as $s^{\frac{1}{6}}$ when s is small. (It also follows that the coefficients v_n in (38) tend to zero like $n^{-\frac{3}{2}}$, in agreement with the values obtained numerically.)

The angular distributions obtained by these methods are plotted in figure 7, with the corresponding curves for the case of isotropic scattering ($p(\cos\Theta) \equiv 1$), from Chandrasekhar (1950). For a slab of sufficient thickness, the transmitted radiation behaves like that from a semi-infinite region, problem (*b*), and the distribution of reflected radiation can be obtained by subtracting the appropriate multiple of curve (*b*) from curve (*a*).

If γ is not zero (so that energy is now absorbed) but small, the fractional change in these distributions is of order k_0 ($\doteqdot \surd(6\gamma)$), and the change could be calculated to this order without much trouble. But it is the asymptotic forms for s large that are changed most, becoming proportional to $g_0(\mu)\,e^{-k_0 s}$ in problem (*a*) and to $g_0(-\mu)\,e^{k_0 s} + Cg_0(\mu)\,e^{-k_0 s}$ in problem (*b*).

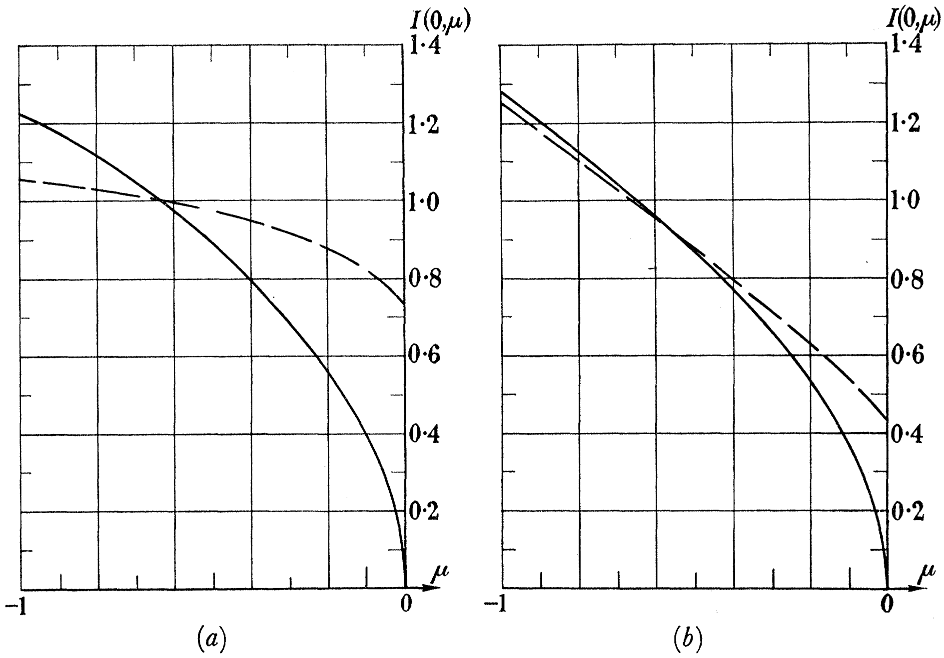

FIGURE 7. Angular distribution of emergent radiation. (*a*) Unidirectional beam (with $F = 1$) incident normally. No net flux; (*b*) constant net flux ($F = 1$) from infinity. No incident radiation. Solid line —— small scattering angle (present theory). Broken line – – – isotropic scattering (Chandrasekhar). All curves refer to conservative case ($\gamma = 0$).

16. Problem of oblique incidence

We now relax the requirement of axial symmetry, so that the intensity depends also on the azimuthal angle ϕ. This extension does not create any great difficulties, because if $I(x, \mu, \phi)$ is expressed as a Fourier series in ϕ

$$I(x, \mu, \phi) = \sum_0^\infty I_m(x, \mu) \cos(m\phi + \epsilon_m),$$

each coefficient I_m satisfies an equation similar to (19).

$$\left(\mu \frac{\partial}{\partial x} + \beta\right) I_m(x, \mu) = \tfrac{1}{2}\beta \int_{-1}^{1} I_m(x, \mu')\, K_m(\mu, \mu')\, d\mu', \tag{41}$$

where
$$K_m(\mu, \mu') = \sum_m^\infty \frac{(n-m)!}{(n+m)!} \varpi_n P_n^{(m)}(\mu)\, P_n^{(m)}(\mu').$$

The boundary conditions on a surface can also be expressed as Fourier series in ϕ, and all the I_m tend to zero at infinity, except I_0 for which the possible asymptotic forms are the same as before. Then the calculation of each I_m involves no new problems.

With a small angle of scattering we find approximate equations corresponding to (32):

$$\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial I_m}{\partial\mu}-\frac{m^2}{1-\mu^2}I_m=\mu\frac{\partial I_m}{\partial s}+\gamma I_m. \tag{42}$$

The whole of § 15 applies to the determination of the component I_0 which is independent of azimuth, except that the incident beam is not normal to the surface. And the other I_m can be calculated similarly in terms of eigenfunctions $g_{mn}(\pm\mu)$, with eigenvalues $\pm k_{mn}$.

The smallest positive eigenvalue, k_{m1}, is about 7·07 for $m=1$, and increases approximately as m^2. Hence the approach of the complete solution to the asymptotic form for large s is in general dominated by the term

$$g_{11}(\mu)\exp[-7{\cdot}07s]\cos\phi=g_{11}(\mu)\exp[-1{\cdot}77\beta\alpha^2x]\cos\phi.$$

Assuming a parallel incident beam carrying net flux πF in direction μ_0, we can find the components I_m as series similar to (38 *a*), and hence give series for the angular distribution of reflected radiation. However, at this point we can make use of the general principles of invariance for the angular distribution, as given in chapter IV of Chandrasekhar (1950). The integrals occurring in the mathematical formulation of these principles can be approximated, for small angles of scattering, as in § 14, and substitution of the series of eigenfunctions leads to relations between the coefficients, greatly reducing the number which must be calculated independently. The resulting solutions, for $-1\leqslant\mu\leqslant 0$, $0\leqslant\mu_0\leqslant 1$, are

$$\left.\begin{aligned}I_0(0,\mu)&=\tfrac{1}{2}F\mu_0\left\{-\tfrac{3}{2}\mu+\sum_1^\infty g_n(-\mu_0)\,g_n(-\mu)+\sum_0^\infty\sum_0^\infty z_{np}g_n(-\mu_0)\,g_p(\mu)\right\},\\ I_m(0,\mu)&=F\mu_0\left\{\sum_{n=1}^\infty g_{mn}(-\mu_0)\,g_{mn}(-\mu)+\sum_{n=1}^\infty\sum_{p=1}^\infty z_{mnp}g_{mn}(-\mu_0)\,g_{mp}(\mu)\right\},\end{aligned}\right\} \tag{43}$$

where $z_{0n}=z_{n0}=v_n \quad (n=0,1,2,\ldots),$

$$\left.\begin{aligned}z_{np}&=-\frac{k_nk_pv_nv_p}{3(k_n+k_p)}\\ z_{mnp}&=-\frac{y_{mn}y_{mp}}{3(k_{mn}+k_{mp})}\end{aligned}\right\}\quad(n=1,2,\ldots;\ p=1,2,\ldots).$$

The v_n are the same as those used in equations (38). The y_{mp} can be found from the relation

$$\sum_{p=1}^\infty z_{mnp}g_{mp}(\mu)=-g_{mn}(-\mu)\quad\text{in}\quad 0\leqslant\mu\leqslant 1,$$

taking any particular value for n.

The same similarity solution (40) should apply to all the components I_m.

17. Further comments on the general steady-state problem

We may now consider the steady-state problem when the scattering angle is not small enough for the approximate equations (32) and (42) to be used, to see how far the methods of §§ 15 and 16 can be applied.

Separation of variables in equation (19) leads to

$$I(x,\mu)=\exp[-\tfrac{1}{4}k\alpha^2\beta x]\,G(\mu),$$

where
$$(1-\tfrac{1}{4}k\alpha^2\mu)\,G(\mu) = \frac{1}{2}\int_{-1}^{1} G(\mu')\,K(\mu,\mu')\,\mathrm{d}\mu'. \tag{44}$$

From this, for many types of scattering function, eigenvalues of k can be found as the zeros of a continued fraction like (36)

$$1-\frac{\rho_1 k^2}{1-}\,\frac{\rho_2 k^2}{1-}\cdots,$$

where
$$\rho_n = \frac{(\frac{1}{4}\alpha^2 n)^2}{(2n-1-\varpi_{n-1})\,(2n+1-\varpi_n)}.$$

This fraction has zeros and poles alternating from 0 to $4/\alpha^2$ (and by symmetry from 0 to $-4/\alpha^2$). It has essential singularities at $\pm 4/\alpha^2$, and the complex k-plane must be cut all along the real axis except for the interval between these two points. It does not appear that there are zeros or singularities off the real axis.

If the scattering function is a finite sum of Legendre polynomials, so that $\varpi_n = 0$ if $n > N$, there are only a finite number of eigenvalues of (44); in other cases there can be an infinite number having $\pm 4/\alpha^2$ as limit points. Thus there are certainly some scattering functions for which the set of eigenfunctions is not a complete set, and it seems very probable that this is always true. But this set must become complete asymptotically as the r.m.s. scattering angle, α, tends to zero, if the assumption in § 15 about the completeness of the set of eigenfunctions of equation (35) is correct (and the assumption is supported by the consistency of the results obtained in §§ 15 and 16). It seems that this state of affairs can be interpreted as follows.

Referring to conservative problems, with axial symmetry, we suppose that if $I(0,\mu)$ is given as a continuous function in $0 < \mu \leqslant 1$, and $I(x,\mu) - \frac{3}{4}F\{-\mu + \beta x(1-\frac{1}{3}\varpi_1)\}$, where F is given, tends to a constant value as x tends to infinity, then $I(x,\mu)$ is determined, doubly-continuous in $-1 \leqslant \mu \leqslant 1$, $0 \leqslant x$, apart from a finite discontinuity at $\mu = x = 0$. We can then take the Laplace transform of equation (19) with respect to $s = \frac{1}{4}\alpha^2\beta x$, and make use of a consequence of the above supposition: that the transform $\mathscr{L}I(\lambda,\mu)$ of $I(x,\mu)$ is a regular function of the complex variable λ, and a continuous function of μ, in $\mathscr{R}\lambda \geqslant 0$, $-1 \leqslant \mu \leqslant 1$, apart from a double pole at $\lambda = 0$. It easily follows that

$$I(0,\mu) = \frac{\beta}{-2\mu}\int_{-1}^{1} \mathscr{L}I\left(\frac{4}{-\alpha^2\mu},\mu'\right) K(\mu,\mu')\,\mathrm{d}\mu' \quad \text{for} \quad -1 \leqslant \mu < 0,$$

and, in cases where the scattering function is a finite sum of Legendre polynomials, that the only singularities of $\mathscr{L}I(\lambda,\mu)$ in the λ-plane cut from $-\infty$ to $-4/\alpha^2$ are a logarithmic singularity at $-4/\alpha^2$, and poles at those eigenvalues of equation (44) which lie in the left half-plane (including the one at the origin).

The Laplace transform is inverted by contour integration

$$I(x,\mu) = \frac{1}{2\pi \mathrm{i}}\int \mathrm{e}^{\lambda s}\,\mathscr{L}I(\lambda,\mu)\,\mathrm{d}\lambda,$$

where the path of integration can be taken to start at $-\infty$ below the real axis and to end at $-\infty$ above the real axis, enclosing all singularities of the integrand. Each pole of the integrand therefore contributes a term of the form $\mathrm{e}^{-ks}\,G(\mu)$ to the solution, and there is also a contribution from the integral around the singularity at $-4/\alpha^2$, which decays with x as $\mathrm{e}^{-\beta x}$, faster than any of the terms contributed by the poles.

If we write the intensity as a function of μ and $s = \frac{1}{4}\alpha^2\beta x$, and let α tend to zero, this term becomes negligible except at $s = 0$, and even there it should become negligible except at discontinuities of $I(0, \mu)$ or $(\partial/\partial\mu)\, I(0, \mu)$. Thus the solution is given by a series of eigenfunctions.

The same thing could be done for any of the components I_m when axial symmetry is not present, and also for non-conservative problems.

Now if the scattering angle is fairly small, the first few eigenvalues and functions will be nearly the same as those obtained for the approximate equations (32) and (42); they can, moreover, be calculated numerically, without difficulty, though simple asymptotic expansions for larger eigenvalues do not appear to be available. The coefficients in the series solution, for given boundary conditions, cannot be found without more difficulty, but again the first few should not differ greatly from those obtained for the approximate equation. So a certain amount of information about the solution can be obtained in this way, especially the manner of approach to the asymptotic form far from the surface, and the distribution of emergent radiation in directions which are not nearly parallel to the surface.

The similarity solution (40) for small values of μ and s cannot be expected to hold unless $\frac{1}{4}\alpha^2 \ll |\mu| \ll 1$, and there is actually a discontinuity, by an amount of order α, in $I(0, \mu)$ at $\mu = 0$.

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References

Batchelor, G. K. 1955 *Tech. Rep. Sch. Elec. Engng, Cornell Univ.* no. 26.
Batchelor, G. K. 1957 *Naval hydrodynamics.* Publication 515, Nat. Acad. Sci., Nat. Res. Coun. p. 409.
Batchelor, G. K. 1959 *J. Fluid Mech.* **5**, 113.
Booker, H. G. & Gordon, W. E. 1950 *Proc. Inst. Radio Engrs, N.Y.*, **38**, 401.
Booker, H. G., Ratcliffe, J. A. & Shinn, D. H. 1950 *Phil. Trans.* A, **262**, 579.
Breitenberger, E. 1959 *Proc. Roy. Soc.* A, **250**, 514.
Chandrasekhar, S. 1950 *Radiative transfer.* Oxford: Clarendon Press.
Ellison, T. H. 1951 *J. Atmos. Terr. Phys.* **2**, 14.
Fejer, J. A. 1953 *Proc. Roy. Soc.* A, **220**, 455.
Hewish, A. 1951 *Proc. Roy. Soc.* A, **209**, 81.
Kraichnan, R. H. 1953 *J. Acoust. Soc. Amer.* **25**, 1096.
Lighthill, M. J. 1953 *Proc. Camb. Phil. Soc.* **49**, 531.
Mintzer, D. 1953*a* *J. Acoust. Soc. Amer.* **25**, 922.
Mintzer, D. 1953*b* *J. Acoust. Soc. Amer.* **25**, 1107.
Mintzer, D. 1954*c* *J. Acoust. Soc. Amer.* **26**, 186.
Obukhoff, A. M. 1943 *C.R. Acad. Sci., U.R.S.S.*, **39**, 46.
Olver, F. 1954 *Phil. Trans.* A, **247**, 307.
Silverman, R. A. 1955 *Proc. Inst. Radio Engrs, N.Y.*, **43**, 1253.
Silverman, R. A. 1957 *J. Appl. Phys.* **28**, 506.
Silverman, R. A. 1958 *Proc. Camb. Phil. Soc.* **54**, 530.
Ueno, S. 1958 *J. Math. Mech.* **7**, 629.
Villars, F. & Weisskopf, V. F. 1955 *Proc. Inst. Radio Engrs, N.Y.*, **43**, 1232.